THE THEORY OF ELECTRICAL ARTIFICIAL LINES AND FILTERS

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BY

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PREFACE.

In modern telegraphy and telephony, both wired and wireless, repeated networks such as artificial transmission lines, line balances, filters, and phase shifters are becoming of increasing importance, and a knowledge of their action is becoming essential to all engineers.

The present book is an attempt to give a general introduction to the theory of such networks and to put the reader in a position to enter into the literature of the subject. The first chapter deals with the theory of the T and Π section artificial lines in considerable detail. Chapter II. gives the general theory of repeated networks and deals with the most general type of artificial line; Chapter III. deals with some generalisations of the T and Π sections.

These three chapters are the most important; the remaining chapters are extensions or applications. The chapter on Filters is restricted to a treatment of the simpler types; the important work of Zobel on composite filters and terminations is omitted, as there are a number of books available which deal with the subject in great detail.

Chapter VIII. deals at some length with the Homographic Transformation, for, though many circle diagrams are in use, the fundamental theory does not appear to be widely known.

Chapter IX. gives a theory of the Multistage Amplifier depending on a slight extension of the methods of Chapter II.

The literature of this subject, which owes so much to Kennelly and G. A. Campbell, is now so vast that no attempt has been made to give a Bibliography.

I wish in conclusion to thank Mr. J. F. Ramsay for finally transcribing and checking the whole MSS. ; without his assistance it would never have been completed.

I wish also to thank Mr. N. R. Bligh for his valuaable help in reading proofs, and particularly for drawing my attention to the equivalent valve circuit shown in Fig. 148(*b*) and to its advantages.

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January, 1930.

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CHAPTER I.

THEORY OF T AND Π SECTION ARTIFICIAL LINES.

§ 1. Definitions.

An *artificial line* will be taken throughout as being an electrical network constructed of any number of identical sections, each section consisting of a passive network. By passive is meant that the network consists of resistances, capacities, self and mutual inductances only, and contains *no* sources of power.

Each section has two pairs of terminals, a pair for "input" and a pair for "output": in addition it must be electrically symmetrical with respect to its two pairs of terminals, i.e. it is immaterial which of the two pairs of terminals is used for input or which for output.

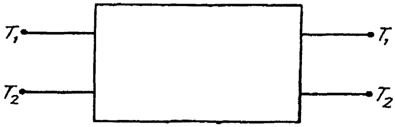


FIG. 1.

The sections are connected so that the output terminals of the first section are connected to the input terminals of the second section, and so on, while the output terminals of the last section are usually closed through a terminating impedance.

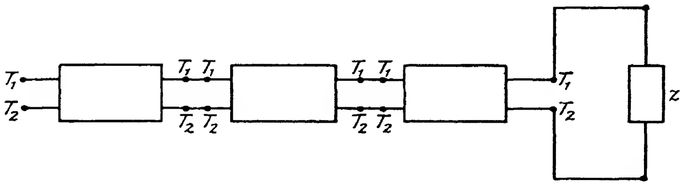


FIG. 2.

Thus an artificial line section may be represented, as in Fig. 1, by a box having two pairs of terminals. An artificial line of three sections closed through a terminal impedance z is shown in Fig. 2.

Various examples of artificial line sections are shown in Fig. 3,

It will be seen that the above definition, which requires only that the sections are electrically symmetrical and passive, is a very general one; that it includes filters and that it does not require that an artificial line shall have any relation to any physically existent telephone or power transmission line.

An artificial line designed to be equivalent to a telephone or

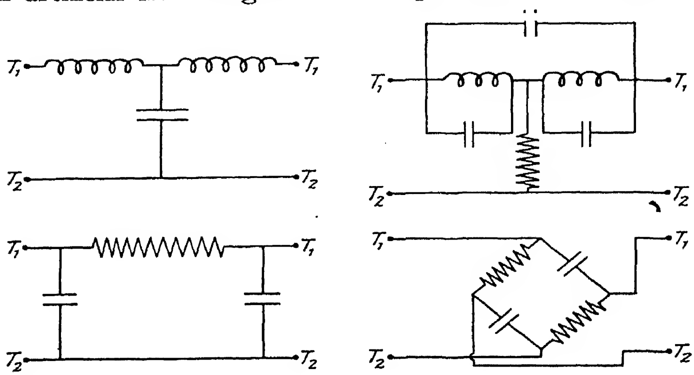


FIG. 3.

power line both as regards input and output terminals will be termed an "Artificial Telephone or Transmission Line."

The term *impedance* will be used throughout in a dual sense, either to mean an electrical network as a physical arrangement of resistances, condensers and inductances having a single pair of terminals, or as the algebraic expression (usually a complex quantity) for the impedance of the network.

§ 2. T and Π Section Artificial Lines.

In this chapter only some of the simpler types will be con-

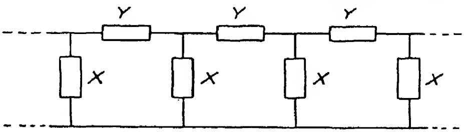


FIG. 4.

sidered, viz. the T and Π sections which both arise from considering the ladder network of which a portion is shown in Fig. 4, having equal series impedances Y, and equal shunt impedances X.

The calculation of current and voltage in such a network is greatly simplified if the terminations are either as shown in Fig. 5(*a*) or 5(*b*).

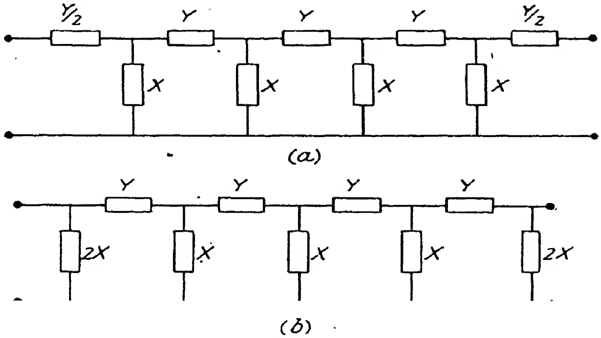


FIG. 5.

The ladder network of Fig. 5(*a*) can be re-drawn as Fig. 6.

It is seen to be a four-section artificial line made up of four sections, such as in Fig. 7.

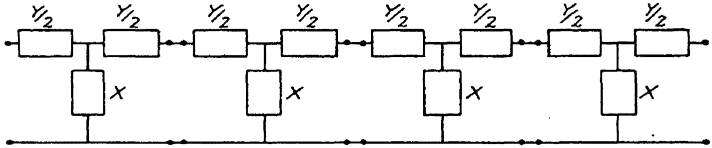


FIG. 6.

Similarly, the network of Fig. 5(*b*) is a four-section artificial line, a section of which is shown in Fig. 8.

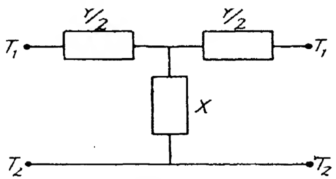


FIG. 7.

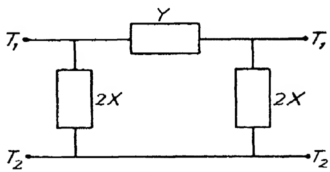
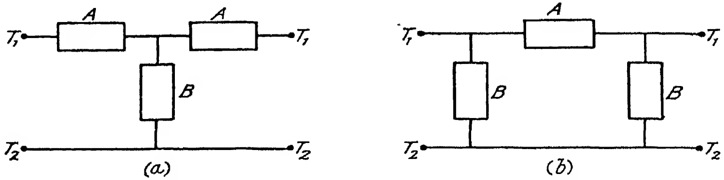


FIG. 8.

The two types of artificial line section shown in Figs. 7 and 8 are known as the T and Π artificial line sections respectively, and are the most common and useful.

In general, however, they will not be regarded as derived from a common ladder network, but will be treated as two separate types, as in Fig. 9.

In each case A is the series impedance and B the shunt impedance.



Standard T Section. Standard Π Section.

FIG. 9.

§ 3. Theory of the T Section Artificial Line—The Difference Equations.

Consider an artificial line (Fig. 10) consisting of n T sections terminated by an impedance z. A voltage V_0 is applied to the input terminals.

Let $V_1, V_2, \ldots V_n$ be the voltages across the output terminals of the first, second, third, etc., sections, so that V_n is the voltage across z.

Let I_0 be the current flowing into the network, and let

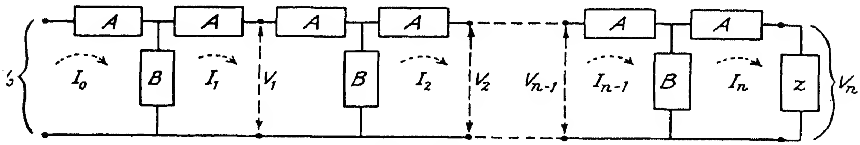


FIG. 10.

$I_1, I_2, \ldots I_n$ be the output currents of the first, second, third, etc., sections, so that I_n is also the current through z.

The problem is: Given A, B, n and V_0, determine $I_0, I_1, \ldots I_n$, and $V_1, V_2, \ldots V_n$.

Consider first one of the intermediate sections, e.g. the rth, together with the $(r+1)$th sections, as shown in Fig. 11. Applying to the central mesh Kirchhoff's Law that the sum of the E.M.F.'s round the mesh is zero we obtain

$$2AI_r + B(I_r - I_{r-1}) + B(I_r - I_{r+1}) = 0$$

i.e. $$I_{r+1} - 2 \,.\, I_r(1 + A/B) + I_{r-1} = 0 \quad . \quad 3.1$$

This is a Difference Equation (cf. Boole, " Finite Differences," Chap. XI.), the solution of which is known to be

$$I_r = a \cosh r\theta + b \sinh r\theta \quad . \quad . \quad . \quad 3.2$$

where θ is given by

$$\cosh \theta = 1 + A/B \quad . \quad . \quad . \quad . \quad 3.3$$

The constants a and b are determined from the boundary conditions.

A similar difference equation can be obtained connecting V_{r+1}, V_r, V_{r-1}. Returning to Fig. 11 we may write down the following set of equations from Kirchhoff's Laws :—

$$\begin{aligned} V_{r-1} &= (A + B)I_{r-1} - BI_r \\ V_r &= (A + B)I_r - BI_{r+1} \\ - V_{r+1} &= (A + B)I_{r+1} - BI_r. \end{aligned}$$

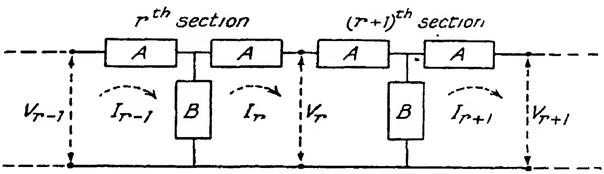


Fig. 11.

Multiply these equations by B, $-$ 2(A + B) and $-$ B respectively and add them, obtaining

$$BV_{r+1} - 2(A + B)V_r + BV_{r-1}$$
$$= (A + B)[BI_{r+1} - 2(A + B)I_r + BI_{r-1}].$$

But the right-hand side is zero by equation 3.1, so that, dividing out by B,

$$V_{r+1} - 2V_r(1 + A/B) + V_{r-1} = 0 \quad . \quad . \quad 3.4$$

a difference equation relating successive V's of exactly the same form as that relating successive I's.

The solution of this equation is

$$V_r = c \cosh r\theta + d \sinh r\theta \quad . \quad . \quad . \quad 3.5$$

where, as before,

$$\cosh \theta = 1 + A/B$$

and c and d are constants determined by the boundary conditions.

§ 4. The Infinite Artificial Line.

The determination of the constants will now be considered for a line having an infinite number of sections. If we assume

that A and B are each or either to a certain extent dissipative, i.e. contain resistance, then both voltage and current in the rth section, as r tends to infinity, will tend to zero.

Thus, in the two equations 3.2 and 3.5, we get, putting $I_r = 0$, $V_r = 0$, and $r \to \infty$,

$$0 = a + b$$
$$0 = c + d$$

since cosh $r\theta$ and sinh $r\theta$ approach equality as $r \to \infty$.

Therefore in general

$$\left.\begin{aligned} I_r &= a(\cosh r\theta - \sinh r\theta) \\ &= ae^{-r\theta} \\ V_r &= c(\cosh r\theta - \sinh r\theta) \\ &= ce^{-r\theta} \end{aligned}\right\} \quad . \quad . \quad . \quad 4.1$$

Of the four constants a, b, c, d, two only, a and c, remain to be determined. By putting $r = 0$ in the last equation, $c = V_0$.

To obtain the value of a consider the first section of the artificial line; from Kirchhoff's Law

$$\begin{aligned} V_0 &= (A + B)I_0 - BI_1 \\ &= (A + B)a - Bae^{-\theta} \quad . \quad . \quad . \quad 4.2 \end{aligned}$$

Since $\cosh \theta = 1 + A/B$, it follows that

$$e^{-\theta} = 1 + \frac{A}{B} - \frac{1}{B}\sqrt{(A^2 + 2AB)} \quad . \quad . \quad 4.3$$

Inserting this value of $e^{-\theta}$ in 4.2,

$$V_0 = a\sqrt{(A^2 + 2AB)}; \quad a = V_0/\sqrt{(A^2 + 2AB)} \quad . \quad 4.4$$

Thus the constants are now completely determined, and the final equations are

$$V_r = V_0 e^{-r\theta} \quad . \quad . \quad . \quad . \quad . \quad 4.5$$
$$I_r = V_0 e^{-r\theta}/\sqrt{(A^2 + 2AB)} \quad . \quad . \quad . \quad 4.6$$

which is the required complete solution.

It is of interest to compare these results with the corresponding results for an infinite uniform transmission line such as given in Fleming's "Propagation of Electric Currents in Telephone and Telegraph Conductors," Chap. III., when it will be seen that they are of the same form.

In the case of the artificial line $n\theta$ corresponds to Pl for the uniform line, and $\sqrt{(A^2 + 2AB)}$ corresponds to $\sqrt{\left(\frac{R + jpL}{S + jpC}\right)}$. A

similar terminology for these quantities will be adopted, viz. θ is the *Propagation Constant per section,* and $\sqrt{(A^2 + 2AB)}$ is the *Characteristic Impedance ;* it will be denoted by Z_0.

Just as in the case of the uniform line the Propagation Constant is a complex quantity, so in the case of the artificial line it will usually be complex, of the form $\alpha + j\beta$.

Here again α is the Attenuation Constant and β the Phase Constant.

Thus $\quad V_r = V_0 e^{-r\theta}$

may be written

$$V_r = V_0 e^{-r(\alpha + j\beta)}$$

$$= V_0 e^{-r\alpha} (\cos r\beta - j \sin r\beta) \quad . \quad . \quad 4.7$$

showing that if a voltage V_0 is applied to the input terminals of the artificial line, the voltage at the output terminals of the rth section will be equal to the input voltage with its amplitude diminished by the factor $e^{-r\alpha}$ to $V_0 e^{-r\alpha}$, and lagging in phase by an angle $r\beta$.

The model described by Fleming (p. 85) applies equally well to the infinite artificial line if the rods are taken as corresponding to voltages at the terminals of successive sections.

The currents are given in a similar way by

$$I_r = \frac{V_0}{Z_0} e^{-r\alpha} (\cos r\beta - j \sin r\beta) \quad . \quad . \quad 4.8$$

and thus only differ from the voltages in that they are multiplied by the factor $1/Z_0$, in general a complex quantity.

The input current to the artificial line is seen to be

$$I_0 = V_0/Z_0 \quad . \quad . \quad . \quad . \quad 4.9$$

so that Z_0 is the impedance of the infinite artificial line measured at its input terminals. The equation for the current may be written

$$I_r = I_0 e^{-r\theta} \quad . \quad . \quad . \quad . \quad 4.10$$

which is of exactly the same form as 4.5.

§ 5. The Finite T Section Line with Terminal Impedance.

The more complicated problem of determining the constants for the case of an artificial line of n sections terminated by an impedance z, as shown in Fig. 10, has now to be treated.

Consider first the current equation

$$I_r = a \cosh r\theta + b \sinh r\theta \quad . \quad . \quad . \quad 5.1$$

The terminal condition for the sending end is the same as was used for the infinite line, viz.

$$I_0(A + B) - BI_1 = V_0.$$

Substituting for I_0 and I_1 from 5.1.

$$(A + B)a - B(a \cosh \theta + b \sinh \theta) = V_0 \quad . \quad 5.2$$

$$\left.\begin{array}{ll} \text{Now} & \cosh \theta = 1 + A/B \\ \text{and therefore} & \sinh \theta = \sqrt{(A^2 + 2AB)}/B = Z_0/B \end{array}\right\} \quad . \quad . \quad 5.3$$

Therefore 5.2 becomes

$$(A + B)a - B\{a(1 + A/B) + b\sqrt{(A^2 + 2AB)}/B\} = V_0$$

whence

$$b = - V_0/Z_0 \quad . \quad . \quad . \quad . \quad 5.4$$

The other constant a may be determined from the application of Kirchhoff's Law to the last mesh of Fig. 10.

$$I_n(A + B + z) - I_{n-1}B = 0 \quad . \quad . \quad . \quad 5{\cdot}5$$

whence, substituting from 5.1 for I_n and I_{n-1} and using the value of b just determined,

$$\left(a \cosh n\theta - \frac{V_0}{Z_0} \sinh n\theta\right)(A + B + z)$$
$$= B\left(a \cosh (n - 1)\theta - \frac{V_0 \sinh (n - 1)\theta}{Z_0}\right) \quad 5.6$$

Before dealing with this equation we should note the following results, which are readily derived from equations 5.3 :—

$$\left.\begin{array}{r} A + B = Z_0 \coth \theta \\ B = Z_0/\sinh \theta \\ A = Z_0 \tanh \theta/2 \end{array}\right\} \quad . \quad . \quad . \quad 5.7$$

Let

$$z = Z_0 \tanh \gamma \quad . \quad . \quad . \quad . \quad 5.8$$

Returning to equation 5.6,

$$a = \frac{V_0}{Z_0} \cdot \frac{(A + B + z) \sinh n\theta - B \sinh (n - 1)\theta}{(A + B + z) \cosh n\theta - B \cosh (n - 1)\theta} \quad . \quad 5.9$$

Now in virtue of 5.7 and 5.8 the denominator of the fraction on the right is equal to

$$Z_0\left\{\cosh n\theta \tanh \gamma + \cosh n\theta \coth \theta - \frac{\cosh (n - 1)\theta}{\sinh \theta}\right\}$$
$$= Z_0\left\{\cosh n\theta \tanh \gamma + \frac{\cosh n\theta \cosh \theta - \cosh (n - 1)\theta}{\sinh \theta}\right\}$$
$$= Z_0\{\cosh n\theta \tanh \gamma + \sinh n\theta\}$$
$$= Z_0 \frac{\sinh (n\theta + \gamma)}{\cosh \gamma} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad 5.10$$

In a similar way the numerator is found to be equal to

$$Z_0 \frac{\cosh (n\theta + \gamma)}{\cosh \gamma} \quad . \quad . \quad . \quad 5.11$$

Substituting these values in 5.9,

$$a = \frac{V_0}{Z_0} \coth (n\theta + \gamma) \quad . \quad . \quad . \quad 5.12$$

The complete solution for the current I_r is thus

$$I_r = \frac{V_0}{Z_0}\{\cosh r\theta \coth (n\theta + \gamma) - \sinh r\theta\}$$

$$= \frac{V_0}{Z_0} \frac{\cosh (\overline{n - r}.\theta + \gamma)}{\sinh (n\theta + \gamma)} \quad . \quad . \quad . \quad . \quad 5.13$$

Now consider the voltage equation

$$V_r = c \cosh r\theta + d \sinh r\theta.$$

Since V_0 is known we have at once, by putting $r = 0$,

$$c = V_0 . \quad . \quad . \quad . \quad . \quad 5.14$$

The other constant is determined by considering the receiving-end termination; by 5.13 I_n is known: also $I_n . z$, the voltage across z, is equal to V_n. Hence,

$$V_0 \cosh n\theta + d \sinh n\theta = V_n = I_n z$$

$$= \frac{V_0}{Z_0} \frac{\cosh \gamma}{\sinh (n\theta + \gamma)} . Z_0 \tanh \gamma$$

$$= \frac{V_0 \sinh \gamma}{\sinh (n\theta + \gamma)} \quad . \quad . \quad . \quad . \quad . \quad 5.15$$

Thus

$$d = \frac{V_0}{\sinh n\theta} . \left\{\frac{\sinh \gamma}{\sinh (n\theta + \gamma)} - \cosh n\theta\right\} = -V_0 \coth (n\theta + \gamma) \quad 5.16$$

Hence both constants are determined, and therefore

$$V_r = V_0\{\cosh r\theta - \coth (n\theta + \gamma) \sinh r\theta\}$$

$$= V_0 . \frac{\sinh(\overline{n - r} . \theta + \gamma)}{\sinh (n\theta + \gamma)} \quad . \quad . \quad . \quad 5.17$$

This equation, together with equation 5.13, gives the complete solution of the problem.

Reproducing them together, we have

$$\left.\begin{aligned} I_r &= \frac{V_0}{Z_0} . \frac{\cosh (\overline{n - r} . \theta + \gamma)}{\sinh (n\theta + \gamma)} \\ V_r &= V_0 . \frac{\sinh (\overline{n - r} . \theta + \gamma)}{\sinh (n\theta + \gamma)} \end{aligned}\right\} \quad . \quad . \quad 5.18$$

or alternatively, introducing z directly by putting $\tanh\gamma = z/Z_0$ the solution is

$$\left.\begin{aligned} I_r &= \frac{V_0}{Z_0}\cdot\frac{Z_0\cosh(n-r)\theta + z\sinh(n-r)\theta}{Z_0\sinh n\theta + z\cosh n\theta} \\ V_r &= V_0\cdot\frac{Z_0\sinh(n-r)\theta + z\cosh(n-r)\theta}{Z_0\sinh n\theta + z\cosh n\theta} \end{aligned}\right\} \quad . \quad 5.19$$

§ 6. The Finite Artificial Line with Terminal Impedance.

Sending-End and Receiving-End Impedances.—The results of the previous section give the complete current and voltage distribution throughout the network, but in general we are more concerned with the *terminal* currents and voltages than with the currents and voltages in intermediate sections.

The current entering the first section of the line, i.e. the sending-end current, is obtained by putting $r = 0$.

Therefore

$$I_0 = \frac{V_0}{Z_0}\coth(n\theta+\gamma) \quad . \quad . \quad . \quad . \quad 6.1$$

$$= \frac{V_0}{Z_0}\cdot\frac{Z_0\cosh n\theta + z\sinh n\theta}{Z_0\sinh n\theta + z\cosh n\theta} \quad . \quad . \quad 6.2$$

Thus, if the sending-end impedance of an artificial line of n sections terminated by an impedance z be denoted by z_n,

$$\left.\begin{aligned} z_n &= Z_0\tanh(n\theta+\gamma) \\ &= Z_0\cdot\frac{Z_0\sinh n\theta + z\cosh n\theta}{Z_0\cosh n\theta + z\sinh n\theta} \end{aligned}\right\} \quad . \quad . \quad 6.3$$

These two results may be compared with the corresponding results for the uniform line. (Fleming, "Propagation of Electric Currents," Chap. III., equations [61].)

The output current is obtained by putting $r = n$, hence

$$I_n = \frac{V_0}{Z_0}\cdot\frac{\cosh\gamma}{\sinh(n\theta+\gamma)}$$

$$= \frac{V_0}{Z_0\sinh n\theta + z\cosh n\theta} \quad . \quad . \quad . \quad 6.4$$

V_n, the voltage across z, is given by

$$V_n = V_0\cdot\frac{\sinh\gamma}{\sinh(n\theta+\gamma)} \quad . \quad . \quad . \quad 6.5$$

$$= z \times I_n.$$

The receiving-end impedance y_n, defined by $\mathbf{I}_n = \mathrm{V}_0/y_n$, is therefore given by

$$y_n = Z_0 \frac{\sinh (n\theta + \gamma)}{\cosh \gamma}$$
$$= Z_0 \sinh n\theta + z \cosh n\theta \quad . \quad . \quad . \quad 6.6$$

It has already been seen that Z_0 is the sending-end impedance of the infinite artificial line, assuming that dissipation is present. There is another important and almost self-evident aspect of Z_0; suppose a line of n T sections is terminated by an impedance equal to Z_0; then from 6.3

$$z_n = Z_0.$$

Further, putting $z = Z_0$ in 5.19,

$$I_r = \frac{V_0}{Z_0} e^{-r\theta},$$
$$V_r = V_0 e^{-r\theta}.$$

Now, these are the same currents and voltages which obtain if the line is infinite; hence, in an artificial line terminated by an impedance equal to Z_0 the current and voltage distributions are exactly the same as if the line were infinite.

§ 7. The Finite Artificial Line Short-circuited at the Output Terminals.

If $z = 0$, i.e. the output terminals are short-circuited, then from equations 5.8, 6.3 and 6.6

$$\gamma = 0$$
$$z_n = Z_0 \tanh n\theta \quad . \quad . \quad . \quad . \quad 7.1$$
$$y_n = Z_0 \sinh n\theta \quad . \quad . \quad . \quad . \quad 7.2$$

§ 8. The Finite Artificial Line Open-circuited at the Output Terminals.

If in equation 6.3 we put $z = \infty$,

$$z_n = Z_0 \coth n\theta \quad . \quad . \quad . \quad . \quad 8.1$$

while from equation 5.19 the terminal voltage is given by

$$V_n = V_0 \operatorname{sech} n\theta \quad . \quad . \quad . \quad . \quad 8.2$$

An important property of the T section artificial line now becomes evident; if Z_c is the sending-end impedance of an artificial line of n sections short-circuited or closed at the output terminals, and Z_f is the sending-end impedance of the same line open-circuited or free at the output terminals, then

$$Z_c = Z_0 \tanh n\theta \quad . \quad . \quad . \quad . \quad 8.3$$
$$Z_f = Z_0 \coth n\theta \quad . \quad . \quad . \quad . \quad 8.4$$

and $\therefore$ $$Z_c Z_f = Z_0^2 \quad . \quad . \quad . \quad . \quad . \quad 8.5$$

This important property, which corresponds to the similar result for the uniform line, enables Z_0 and θ to be determined for any given T section at any frequency by means of two direct impedance measurements. In the next chapter it will be shown that this is true for any symmetrical network, and it will be taken as a fundamental property of the general artificial line.

§ 9. The *Π* Section Artificial Line.

The theory of the Π section artificial line can be dealt with in a very similar manner.

As the difference equations are of the same form it is unnecessary to go through the details. The only difference lies in the characteristic impedance, which in this case is given by

$$Z_0 = \frac{AB}{\sqrt{(A^2 + 2AB)}},$$

while $\cosh \theta = 1 + A/B$, as for the T section.

With this difference for Z_0 the whole of the formulæ of §§ 4, 5, 6, 7 and 8 can be taken over bodily.

§ 10. Simple Networks related to the T and *Π* Section Lines.

Taking first the T section line, we have

$$Z_0 = \sqrt{(A^2 + 2AB)}$$

and

$$\cosh \theta = 1 + A/B.$$

It has already been noticed (5.7) that

$$\left.\begin{aligned} A &= Z_0 \tanh \theta/2 \\ A + B &= Z_0 \coth \theta \end{aligned}\right\} \quad . \quad . \quad . \quad 10.1$$

In fact, this last result is nothing more or less than equation 8.1 applied to a single T section.

By 7.1 we see that, for a single section,

$$Z_c = Z_0 \tanh \theta.$$

Thus, from first principles,

$$Z_0 \tanh \theta = A + \frac{1}{\dfrac{1}{A} + \dfrac{1}{B}} = A + \frac{AB}{A + B} \quad . \quad 10.2$$

and further, since

$$A = Z_0 \tanh \theta/2,$$

$$\begin{aligned} Z_0 \coth \theta/2 &= \frac{Z_0^2}{Z_0 \tanh \theta/2} \\ &= \frac{A^2 + 2AB}{A} \\ &= A + 2B \quad . \quad . \quad . \quad 10.3 \end{aligned}$$

From these we have

$$\begin{aligned} B &= A + B - A \\ &= Z_0 \coth\theta - Z_0 \tanh\theta/2 \\ &= \frac{Z_0}{\sinh\theta} \end{aligned} \qquad 10.4$$

Associated with the T section artificial line we have then a number of simple networks whose impedances are simple functions of Z_0 and θ.

By using these results the T section can be represented as in Fig. 12.

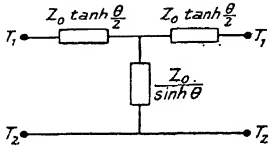


Fig. 12.

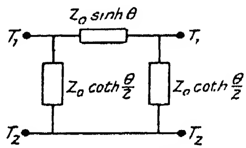


Fig. 13.

Similarly, for the Π section it can be shown that

$$\left.\begin{aligned} A &= Z_0 \sinh\theta \\ B &= Z_0 \coth\theta/2 \end{aligned}\right\} \qquad 10.5$$

$$\frac{AB}{A + 2B} = Z_0 \tanh\theta/2 \qquad 10.6$$

$$\frac{AB}{A + B} = Z_0 \tanh\theta \qquad 10.7$$

$$\frac{B(A + B)}{A + 2B} = Z_0 \coth\theta \qquad 10.8$$

while, corresponding to Fig. 9(*b*) there is Fig. 13.

§ 11. "Equivalent T" and "Equivalent *Π*" for Uniform and Artificial Lines.

It has already been pointed out how similar the artificial line formulæ are to those of the uniform line. Consider the relation between a uniform line of length l, characteristic impedance Z_0, and propagation constant P, per unit length, with the artificial line shown in Fig. 14, an artificial line of one section whose constants are Z_0 and Pl.

It is at once obvious that the artificial line and the uniform line are identical so far as measurements made at the two ends are

concerned. The T section of Fig. 14 is accordingly termed the "equivalent T" section for the uniform line. The exact equivalence, however, holds only for calculation purposes; physically the exact equivalent cannot be constructed.

At any one frequency $Z_0 \tanh Pl/2$ and $Z_0/\sinh Pl$ can be represented by a resistance in series with a condenser or inductance, but the values of these change with frequency.

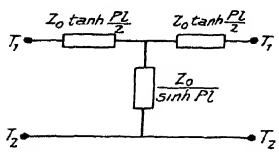


FIG. 14.—Equivalent T for Uniform Line.

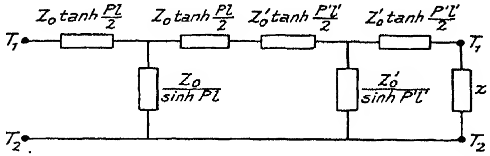


FIG. 15.

The equivalent T is extremely useful for simplifying calculations; e.g. suppose a telephone line consisting of l miles of line and having constants Z_0 and P is followed by l' miles of line having constants Z'_0 and P', terminated by an impedance z, it being required to calculate the sending- and receiving-end impedances.

Having replaced each length of line by its equivalent T, as in Fig. 15, we have only to calculate the input and output impedances

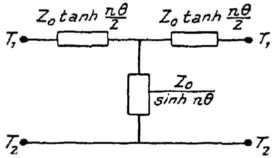


FIG. 16.—Equivalent T for n section Artificial Line.

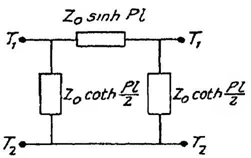


FIG. 17.—Equivalent Π for Uniform Line.

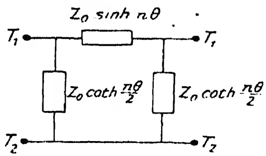


FIG. 18.—Equivalent Π for n section Artificial Line.

of this simple three-mesh network. These can be written down without further appeal to telephonic transmission theory.

Use will be made of the equivalent T in Chapter VI. in dealing with the loaded cable.

In addition, we may use the equivalent T to simplify problems in artificial lines proper; for example, the T section shown in Fig. 16 is equivalent to an artificial line of n sections of constants Z_0 and θ.

Similarly there is the equivalent Π for a length of uniform line and the equivalent Π for an artificial line of n sections; these are shown in Figs. 17 and 18 respectively.

§ 12. Successive Approximations to Z_0 for T and *Π* Section Lines.

It may happen, as will be seen later in Chapter IV., that it is required to construct networks whose impedance approximates to the Z_0 of an artificial line. If the line is dissipative it follows that, since $\tanh n\theta$ and $\coth n\theta$ approach unity as n tends to infinity, any degree of approximation can be obtained by using a sufficient number of sections, and by either short-circuiting the output terminals or by open-circuiting them: the degree of approximation is readily found by evaluating $\coth n\theta$ or $\tanh n\theta$.

There are, however, interesting series of intermediate approximations.

Consider an n section T line having an impedance z across the output terminals. Let the value of this impedance be A, i.e. equal to the series members of the section.

Now $$A = Z_0 \tanh \theta/2,$$
and therefore in 5.8
$$Z_0 \tanh \gamma = z = A = Z_0 \tanh \theta/2,$$
$$\gamma = \theta/2,$$
and then from 6.3
$$z_n = Z_0 \tanh (n\theta + \gamma)$$
$$= Z_0 \tanh (n + \tfrac{1}{2})\theta.$$

Again, consider the same n sections terminated by A + 2B.

Now $$A + 2B = Z_0 \coth \theta/2$$
$$= Z_0 \tanh \left(\frac{\theta}{2} + j\frac{\pi}{2}\right).$$

Then in 5.8,
$$Z_0 \tanh \gamma = z = A + 2B = Z_0 \tanh \left(\frac{\theta}{2} + j\frac{\pi}{2}\right),$$
$$\gamma = \frac{\theta}{2} + j\frac{\pi}{2},$$

and therefore from 6.3, in this case,
$$z_n = Z_0 \tanh (n\theta + \gamma)$$
$$= Z_0 \tanh \left(n\theta + \frac{\theta}{2} + j\frac{\pi}{2}\right)$$
$$= Z_0 \coth (n + \tfrac{1}{2})\theta.$$

Two series of successive approximations, viz.,

$$Z_0 \tanh m\theta/2,$$
$$Z_0 \coth m\theta/2,$$

where $m = 1, 2, 3, 4 \ldots$, can therefore be constructed for Z_0.

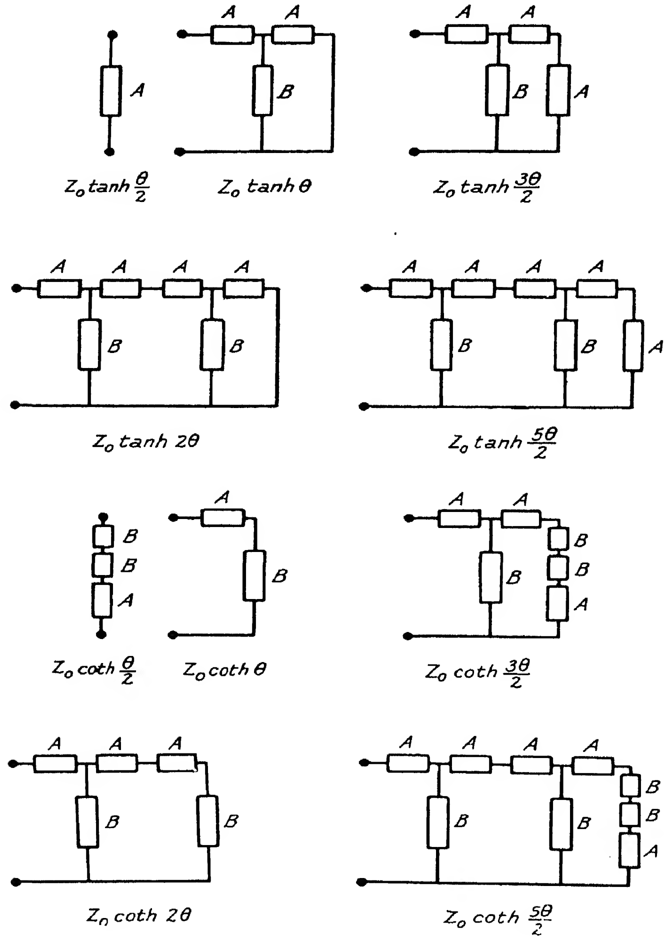


FIG. 19.—The two series of approximations to Z_0 for a T Section Artificial Line.

Figs. 19 and 20 show the first five members of each series for the T section and the Π section line respectively.

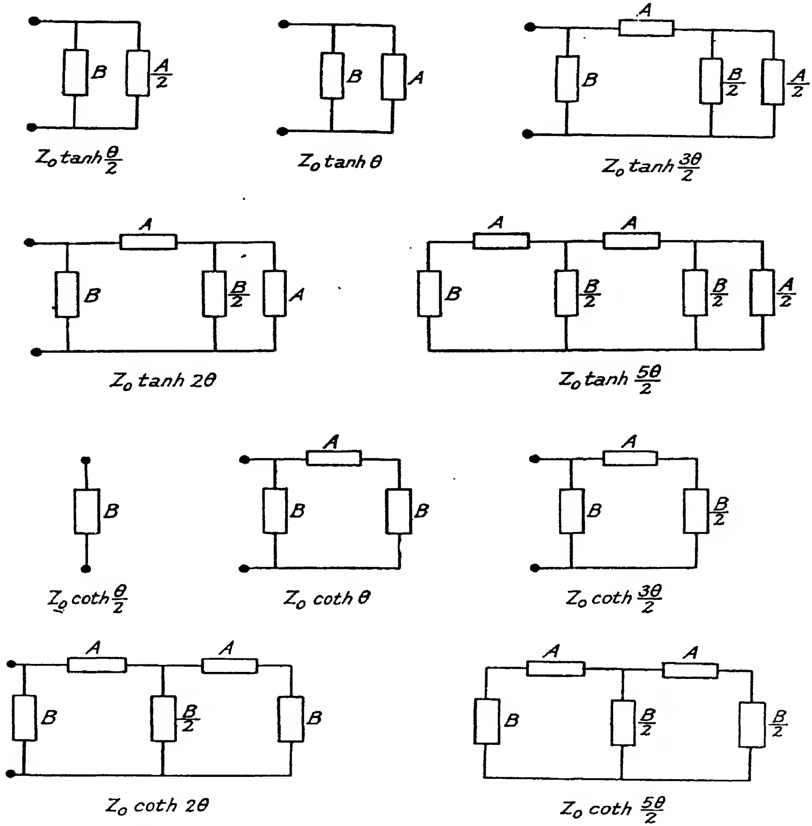


FIG. 20.—The two series of approximations to Z_0 for a Π Section Line.

EXAMPLES.

1. In a T section artificial line

$$\sinh\frac{\theta}{2} = \sqrt{\frac{A}{2B}}.$$

Find the corresponding result for a Π section line.

2. To show how the theory can be applied to a practical case, let us consider the T section shown in Fig. 21.

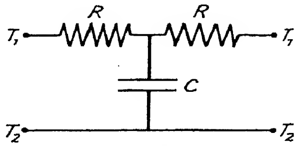


FIG. 21.

We have seen that, for a T section,

$$Z_0 = \sqrt{(A^2 + 2AB)}$$

and

$$\cosh \theta = 1 + A/B.$$

In this case

$$A = R$$
$$B = \frac{1}{j\omega C}.$$
$$\therefore \quad Z_0 = \sqrt{\left(R^2 + \frac{2R}{j\omega C}\right)}$$
$$\cosh \theta = 1 + jw CR.$$

Let

$$R = 1000 \text{ ohms},$$
$$C = 1 \text{ microfarad},$$
$$\omega = 2000 \text{ radians per second}.$$

Then

$$\begin{aligned} Z_0 &= \sqrt{\left\{10^6 + \frac{2 \times 10^3}{j \times 2 \times 10^3 \times 10^{-6}}\right\}} \\ &= 10^3\sqrt{(1 - j)} \\ &= 1190 \ \underline{/-22\tfrac{1}{2}^\circ} \end{aligned}$$

$$\begin{aligned} \cosh \theta &= 1 + j\omega CR \\ &= 1 + j \times 2 \times 10^3 \times 10^{-6} \times 10^3 \\ &= 1 + j2. \end{aligned}$$

i.e.

$$\cosh (\alpha + j\beta) = 1 + j2.$$

$$\begin{aligned} \therefore \quad \alpha + j\beta &= \cosh^{-1} (1 + j2) \\ &= \cosh^{-1} \frac{2\sqrt{2} \pm 2}{2} + j \cos^{-1} \frac{2\sqrt{2} \mp 2}{2} \\ &= \cosh^{-1} 2{\cdot}414 + j \cos^{-1} 0{\cdot}414 \\ &= 1{\cdot}53 + j\,1{\cdot}14. \end{aligned}$$

$\therefore \alpha = 1{\cdot}53.$

$\beta = 1{\cdot}14.$

3. Show that the T sections of Fig. 22 (i) and (ii) have the same characteristic impedance if a, b and c are numerical quantities satisfying

$$a < 1$$
$$b = \frac{1 - a^2}{2a}$$
$$c = 1/a.$$

[Zobel.]

Similarly the Π sections of Fig. 23 have the same characteristic impedance if

$$a < 1$$
$$b = \frac{2a}{1 - a^2}$$
$$c = 1/a.$$

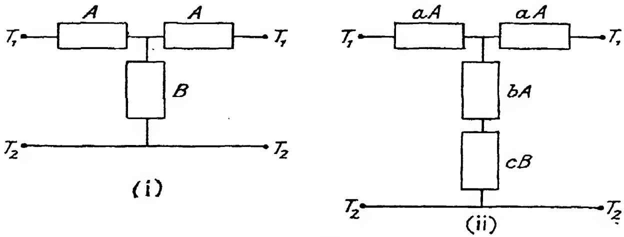


FIG. 22.

4. If V'_r is the voltage across the shunt member of the rth section in the problem treated in § 5, then

$$V_r' = \frac{V_0}{\cosh \theta/2} \cdot \frac{\sinh (\overline{n - r + \tfrac{1}{2}} \cdot \theta + \gamma)}{\sinh (n\theta + \gamma)}.$$

5. If, in the problem treated in § 5, the sections had been Π sections and if I'_r is the current in the series member of the rth section, then

$$I_r' = \frac{V_0}{Z_0 \cosh \theta/2} \cdot \frac{\cosh (\overline{n - r + \tfrac{1}{2}} \cdot \theta + \gamma)}{\sinh (n\theta + \gamma)}.$$

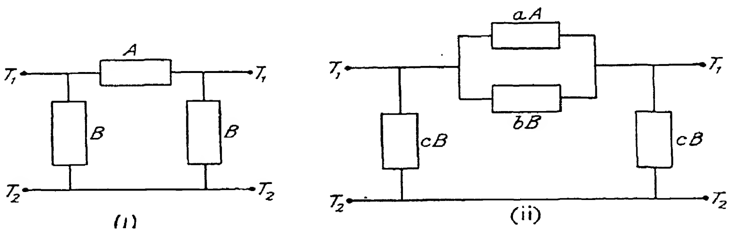


FIG. 23.

6. If, in the problem treated in § 5, instead of applying a voltage V_0 direct to the input terminals, a voltage V in series with an impedance q is applied, then

$$V_r = V \frac{\sinh (\overline{n - r} \cdot \theta + \gamma) \cosh \eta}{\sinh (n\theta + \gamma + \eta)}$$

and

$$I_r = \frac{V}{Z_0} \frac{\cosh (\overline{n - r} \cdot \theta + \gamma) \cosh \eta}{\sinh (n\theta + \gamma + \eta)},$$

where

$$Z_0 \tanh \eta = q.$$

CHAPTER II.

GENERAL THEORY OF ARTIFICIAL LINES.

§ 13. The Theory of Repeated Networks.

In the previous chapter the theory of the T and Π section line has been dealt with in detail; in this chapter the more general artificial line, as defined on page 1, will be considered. However, to make the treatment as wide as possible a more general type of repeated network, in which electrical symmetry is not essential, will first be considered, and then from it, by adding the condition for symmetry, will be deduced the general artificial line theory.

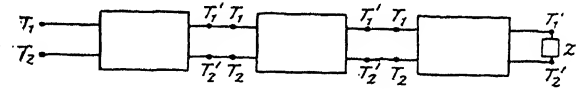


Fig. 24.

Any section of this repeated network may be represented by a box having two pairs of terminals, T_1T_2 and $T_1'T_2'$. The only condition it has to fulfil is that of passivity.

A repeated network of three such sections terminated by an impedance z is shown in Fig. 24, where the pair of terminals T_1T_2 is taken as input.

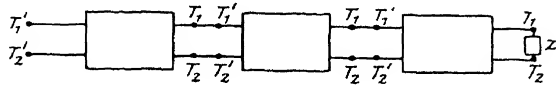


Fig. 25.

Owing to the lack of symmetry it will be necessary, in addition, to consider the case shown in Fig. 25, where $T_1'T_2'$ are taken as input terminals.

The problem considered in the first place may be stated as follows: Given a repeated network of n sections terminated by an impedance z, determine the sending-end and receiving-end imped-

ances, taking either T_1T_2 or $T_1'T_2'$ as input. The values of the currents and voltages at the intermediate junctions need not be considered as it will be seen later that when the solution of the above problem is known their values can be found from it.

It will be shown that the solution can be stated in terms of three constants, and that when the sections are symmetrical the number becomes reduced to two, as in the case of the T and Π section artificial line.

Consider the case of a single network of the more general type, as in Fig. 26, having connected to the terminals T_1T_2 a generator of voltage E_1 through an impedance Q_1, and to the terminals $T_1'T_2'$ a generator of voltage E_2 through an impedance Q_2.

Assign to all the meshes of the network circulating currents $x_1, x_2, \ldots x_m$; x_1 to the mesh containing E_1; x_m to the mesh containing E_2, while $x_2, x_3 \ldots x_{m-1}$ are assigned to meshes inside the network.

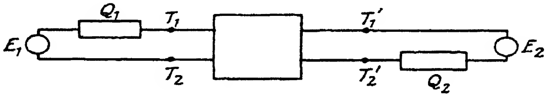


FIG. 26.

The series of mesh equations can now be written down; they will be of the form

$$\left.\begin{array}{lllllll}
(Q_1 + a_{11})x_1 & + a_{12}x_2 & + & \ldots & a_{1m}x_m & = E_1 \\
a_{21}x_1 & + a_{22}x_2 & + & \ldots & a_{2m}x_m & = 0 \\
\cdot & \cdot & \cdot & \ldots & \cdot & = 0 \\
a_{r1}x_1 & + a_{r2}x_2 & + & \ldots & a_{rm}x_m & = 0 \\
\cdot & \cdot & \cdot & \ldots & \cdot & = 0 \\
a_{m1}x_1 & + a_{m2}x_2 & + & \ldots & (Q_2 + a_{mm})x_m & = E_2
\end{array}\right\} \quad . \quad 13.1$$

In these equations Q_1 and Q_2 will occur only once as shown, while all the a's will be functions of the impedances inside the network, and will be independent of Q_1 and Q_2; a_{rr} will be the sum of the impedances taken round the rth mesh of the network, while $-a_{rs} = -a_{sr}$ will be the mutual or common impedances of the rth and sth meshes.

First put $Q_1 = 0$, $Q_2 = z$, $E_1 = E$ and $E_2 = 0$; the input and output currents when a voltage E is applied to T_1T_2 and an impedance z connected to $T_1'T_2'$ (as in Fig. 27(a)) can then be determined.

Solving the equations for x_1 and x_m, the required currents are

$$x_1 = \frac{E\left\{z\dfrac{\partial^2\Delta}{\partial a_{11}\partial a_{mm}} + \dfrac{\partial\Delta}{\partial a_{11}}\right\}}{z\dfrac{\partial\Delta}{\partial a_{mm}} + \Delta} \quad . \quad . \quad . \quad 13.2$$

$$x_m = \frac{E\dfrac{\partial\Delta}{\partial a_{1m}}}{z\dfrac{\partial\Delta}{\partial a_{mm}} + \Delta} \quad . \quad . \quad . \quad . \quad . \quad 13.3$$

where Δ is the determinant

$$\begin{vmatrix} a_{11}, & a_{12}, & \cdots & a_{1m} \\ a_{21}, & a_{22}, & \cdots & a_{2m} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1}, & a_{m2}, & \cdots & a_{mm} \end{vmatrix}$$

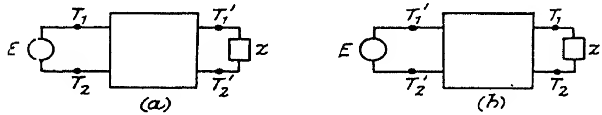


FIG. 27.

and $\dfrac{\partial\Delta}{\partial a_{rs}}$ denotes the determinant, obtained from Δ by deleting the row and column containing a_{rs}, multiplied by $(-1)^{r+s}$; similarly $\dfrac{\partial^2\Delta}{\partial a_{rs}\partial a_{tu}}$ is $(-1)^{r+s+t+u}$ times the determinant obtained by deleting from Δ the rows and columns containing a_{rs} and a_{tu}.

Now put $E_1 = 0$, $E_2 = E$, $Q_1 = z$ and $Q_2 = 0$ and solve for x_m and x_1. These give input and output currents when a voltage E is applied to $T_1'T_2'$, and an impedance z is connected to T_1T_2, as in Fig. 27 (*b*).

Then

$$x_m = \frac{E\left\{z\dfrac{\partial^2\Delta}{\partial a_{11}\partial a_{mm}} + \dfrac{\partial\Delta}{\partial a_{mm}}\right\}}{z\dfrac{\partial\Delta}{\partial a_{11}} + \Delta} \quad . \quad . \quad 13.4$$

$$x_1 = \frac{E\dfrac{\partial\Delta}{\partial a_{m1}}}{z\dfrac{\partial\Delta}{\partial a_{11}} + \Delta} \quad . \quad . \quad . \quad . \quad . \quad 13.5$$

Thus the input and output currents, whichever end is used as sending-end, can be expressed in terms of the five constants

$$\frac{1}{\Delta}\cdot\frac{\partial^2\Delta}{\partial a_{11}\partial a_{mm}};\quad \frac{1}{\Delta}\cdot\frac{\partial\Delta}{\partial a_{11}};\quad \frac{1}{\Delta}\cdot\frac{\partial\Delta}{\partial a_{mm}};\quad \frac{1}{\Delta}\cdot\frac{\partial\Delta}{\partial a_{1m}}$$

$$\text{and}\quad \frac{1}{\Delta}\cdot\frac{\partial\Delta}{\partial a_{m1}}.$$

These constants, however, are not independent, for since Δ is symmetrical about the diagonal $a_{11}a_{mm}$,

$$\frac{\partial\Delta}{\partial a_{1m}} = \frac{\partial\Delta}{\partial a_{m1}} \qquad . \quad . \quad . \quad . \quad 13.6$$

and further,*

$$\Delta\cdot\frac{\partial^2\Delta}{\partial a_{11}\partial a_{mm}} = \frac{\partial\Delta}{\partial a_{11}}\cdot\frac{\partial\Delta}{\partial a_{mm}} - \frac{\partial\Delta}{\partial a_{1m}}\cdot\frac{\partial\Delta}{\partial a_{m1}} \qquad . \quad 13.7$$

Thus in virtue of 13.6 and 13.7, whichever of the pairs of terminals is used as input, both the sending-end and receiving-end impedances can be stated in terms of any three of the constants; the constants can themselves be calculated directly from the known structure of the network.

Considering now the case of a number of networks, let the sending-end impedance of a network such as that shown in Fig. 24, with T_1T_2 as input terminals and terminated by an impedance z, be z_1 if there is one section, z_2 if there are two sections and so on; let z_1', z_2', etc., be the corresponding quantities when $T_1'T_2'$ are taken as input terminals.

Then from 13.2

$$z_1 = \frac{z\dfrac{\partial\Delta}{\partial a_{mm}} + \Delta}{z\dfrac{\partial^2\Delta}{\partial a_{11}\partial a_{mm}} + \dfrac{\partial\Delta}{\partial a_{11}}} \qquad . \quad . \quad . \quad 13.8$$

Thus, as is well known, the input impedance is obtained by transforming the terminal impedance z by a homographic transformation † of the form

$$z_1 = \frac{az + b}{cz + d} \qquad . \quad . \quad . \quad . \quad 13.9$$

The sending-end impedance for two sections terminated by z is the sending-end impedance of a single section terminated by z_1,

* Muir, "Treatise on Determinants" (1882), p. 132.

† The geometrical aspect of the Homographic Transformation is dealt with in detail in Chap. VIII.

and is thus obtained by applying to z two successive applications of the transformation; thus

$$z_2 = \frac{az_1 + b}{cz_1 + d}.$$

Turning now to the homographic transformation

$$z_1 = \frac{az + b}{cz + d},$$

there will, in general, be two different values of z which will be unaltered by the transformation; these will be given by the equation

$$z = \frac{az + b}{cz + d},$$

or

$$cz^2 - (a - d)z - b = 0 \quad . \quad . \quad . \quad 13.10$$

If ξ and ζ are the two roots of this equation

$$\left.\begin{aligned} \xi &= (a - d + \mathrm{M}^{\frac{1}{2}})/(2c) \\ \zeta &= (a - d - \mathrm{M}^{\frac{1}{2}})/(2c) \end{aligned}\right\} \quad . \quad . \quad . \quad 13.11$$

where

$$\mathrm{M} = (d - a)^2 + 4bc.$$

If ξ and ζ are distinct, then

$$\begin{aligned} \frac{z_1 - \xi}{z_1 - \zeta} &= \frac{(az + b) - \xi(cz + d)}{(az + b) - \zeta(cz + d)} \\ &= \frac{(z - \xi)}{(z - \zeta)} \cdot \frac{(a - c\xi)}{(a - c\zeta)} = \mathrm{K} \cdot \frac{z - \xi}{z - \zeta} \quad . \quad 13.12 \end{aligned}$$

where

$$\mathrm{K} = \frac{a - c\xi}{a - c\zeta} = \frac{a + d - \mathrm{M}^{\frac{1}{2}}}{a + d + \mathrm{M}^{\frac{1}{2}}},$$

and is termed the multiplier of the substitution.

The alternative form of the substitution given in 13.12 has the advantage that repeated application is easy. For z_n the variable after n applications of the transformation is obviously given by

$$\frac{z_n - \xi}{z_n - \zeta} = \mathrm{K}^n \cdot \frac{z - \xi}{z - \zeta} \quad . \quad . \quad . \quad 13.13$$

which, if put in the usual form, becomes

$$z_n = \frac{z(\xi - \zeta\mathrm{K}^n) + (\mathrm{K}^n - 1)\xi\zeta}{z(1 - \mathrm{K}^n) + (\xi\mathrm{K}^n - \zeta)} \quad . \quad . \quad 13.14$$

In this put

$$\left.\begin{aligned} \mathrm{K} &= e^{-2\theta} \\ -j\sqrt{\left(\frac{\xi}{\zeta}\right)} &= e^{\phi} \\ j\sqrt{(\xi\zeta)} &= \mathrm{Z} \end{aligned}\right\} \quad . \quad . \quad . \quad 13.15$$

Then

$$z_n = Z\frac{z(e^{n\theta+\phi} + e^{-n\theta-\phi}) + Z(e^{n\theta} - e^{-n\theta})}{z(e^{n\theta} - e^{-n\theta}) + Z(e^{n\theta-\phi} + e^{-n\theta+\phi})}$$

$$= Z\frac{z\cosh(n\theta+\phi) + Z\sinh n\theta}{z\sinh n\theta + Z\cosh(n\theta-\phi)} \quad . \quad . \quad 13.16$$

In this form the repeated transformation involves three constants, Z, θ, and ϕ.

Comparing the new transformation for the case of a single network with its original form given in 13.8, it is seen that

$$Z^2 = \Delta \div \frac{\partial^2\Delta}{\partial a_{11}\partial a_{mm}} \quad . \quad . \quad . \quad 13.17$$

while K, M, ξ and ζ and therefore θ and ϕ can be determined in terms of the impedances of which the network is constructed. So far T_1T_2 have been taken as input terminals; let Z', θ', ϕ', be the corresponding constants when $T_1'T_2'$ is the input: it is readily shown that

$$Z' = Z; \quad \theta' = \theta; \quad \phi' = -\phi,$$

so that, corresponding to 13.16,

$$z_n' = Z\frac{z\cosh(n\theta-\phi) + Z\sinh n\theta}{z\sinh n\theta + Z\cosh(n\theta+\phi)} \quad . \quad . \quad 13.18$$

The actual expression of Z, θ, and ϕ in terms of Δ and its minors has not been given as there is an alternative way of expressing them which is usually more direct and simple. For let Z_c and Z_f be the impedances of a single network measured at the terminals T_1T_2 with $T_1'T_2'$ closed and open-circuited respectively, and let Z_c' and Z_f' be the corresponding quantities measured at $T_1'T_2'$.

Putting $z = 0$ or ∞ and $n = 1$ in 13.16 and 13.18,

$$\left.\begin{aligned} Z_c &= Z\frac{\sinh\theta}{\cosh(\theta-\phi)} \\ Z_f &= Z\frac{\cosh(\theta+\phi)}{\sinh\theta} \\ Z_c' &= Z\frac{\sinh\theta}{\cosh(\theta+\phi)} \\ Z_f' &= Z\frac{\cosh(\theta-\phi)}{\sinh\theta} \end{aligned}\right\} \quad . \quad . \quad 13.19$$

From these it follows that

$$\left.\begin{aligned} Z^2 &= Z_cZ_f' = Z_c'Z_f \\ Z \sinh \phi &= \tfrac{1}{2}(Z_f - Z_f') \\ Z \coth \theta \cosh \phi &= \tfrac{1}{2}(Z_f + Z_f') \end{aligned}\right\} \quad . \quad . \quad 13.20$$

As regards sending-end impedances the problem is now completely solved in terms of Z_c, Z_f, Z_c' and Z_f' or any three of them.

The question of the receiving-end impedances now requires to be treated. With T_1T_2 as input, let y_1, y_2, y_3, etc., be the receiving-end impedance of one, two, three, etc., sections terminated by z, and let y_1', y_2', y_3', etc., be the corresponding quantities with $T_1'T_2'$ as input.

Then

$$\begin{aligned} y_1 &= \left(z\frac{\partial\Delta}{\partial a_{mm}} + \Delta\right) \div \frac{\partial\Delta}{\partial a_{1m}} \\ &= \frac{z\cosh(\theta + \phi) + Z\sinh\theta}{\cosh\phi} \end{aligned} \quad . \quad . \quad 13.21$$

Consider now y_2, the receiving-end impedance of two sections terminated by z; the voltage across the input of the second network will be the terminal voltage of a single network terminated by z_1, and will therefore be

$$Ez_1 \div \frac{z_1\cosh(\theta + \phi) + Z\sinh\theta}{\cosh\phi}.$$

From this, applying 13.21 again, the current through z will be

$$\frac{Ez_1\cosh\phi}{z_1\cosh(\theta + \phi) + Z\sinh\theta} \cdot \frac{\cosh\phi}{z\cosh(\theta + \phi) + Z\sinh\theta}.$$

On substituting for z_1 in terms of z this reduces to

$$\frac{E\cosh\phi}{z\cosh(2\theta + \phi) + Z\sinh 2\theta},$$

whence

$$y_2 = \frac{z\cosh(2\theta + \phi) + Z\sinh 2\theta}{\cosh\phi} \quad . \quad . \quad 13.22$$

Similarly

$$y_n = \frac{z\cosh(n\theta + \phi) + Z\sinh n\theta}{\cosh\phi} \quad . \quad . \quad 13.23$$

and

$$y_n' = \frac{z\cosh(n\theta - \phi) + Z\sinh n\theta}{\cosh\phi} \quad . \quad . \quad 13.24$$

These last two formulæ, together with 13.16 and 13.18, give a complete solution of the problem originally set.

§ 14. The General Artificial Line.

To obtain the general artificial line theory it is necessary to add the symmetry condition. Thus Z_f must be put equal to Z_f' when also Z_c will equal Z_c' and then from equation 13.20

$$\phi = 0$$

Putting $\phi = 0$ in 13.16, 13.18, 13.23 and 13.24 reduces them to the same form as the corresponding equations 6.3 and 6.6 obtained in Chapter I. for the T and Π sections, Z becoming Z_0, the Characteristic Impedance, θ the Propagation Constant.

Corresponding to § 7 of Chapter I. it will be seen that the sending-end impedance of an artificial line section short-circuited at the output is $Z_0 \tanh \theta$, while if open-circuited at the output it is $Z_0 \coth \theta$.

From these two results it is possible to determine the constants.

The current and voltages at the junctions of the sections of the general artificial line will now be determined; it will be shown that they also are given by equations of exactly the same form as those derived for the T and Π section lines.

Take a general artificial line of n sections terminated by an impedance z, and as in Chapter I., let V_0 be the voltage applied at the input and I_0 the input current. Let $V_1, V_2, \ldots V_n$ and $I_1, I_2, \ldots I_n$ be the voltages and currents respectively at the output terminals of the first, second, third, etc., sections.

From Chapter I., 6.4, or from 13.23 after putting $\phi = 0$, we have

$$I_n = \frac{V_0}{Z_0 \sinh n\theta + z \cosh n\theta} \qquad . \quad . \quad 14.1$$

Now V_r, the voltage at the output terminals of the rth section, is also the input voltage to the $(r + 1)$th section. Hence V_r is that voltage which when applied to an artificial line of $(n - r)$ sections terminated by z, gives rise to a current through z equal to I_n. Thus I_n is also equal to

$$\frac{V_r}{Z_0 \sinh (n - r)\theta + z \cosh (n - r)\theta} \qquad . \quad . \quad 14.2$$

Equating the two values of I_n

$$V_r = V_0 \cdot \frac{Z_0 \sinh (n - r)\theta + z \cosh (n - r)\theta}{Z_0 \sinh n\theta + z \cosh n\theta}$$

$$= V_0 \cdot \frac{\sinh (\overline{n - r} \,.\, \theta + \gamma)}{\sinh (n\theta + \gamma)} \qquad . \quad . \quad . \quad . \quad 14.3$$

where $$Z_0 \tanh \gamma = z.$$

This is of the same form as the voltage equation obtained in Chapter I., § 5, eqn. 17.

Similarly

$$I_r = \frac{V_0}{Z_0} \cdot \frac{\cosh\,(\overline{n - r} \,.\, \theta + \gamma)}{\sinh\,(n\theta + \gamma)} \qquad . \qquad . \qquad 14.4$$

is of the same form as the current equation 5.13.

Thus equations 14.3, 14.4 together with 13.16, 13.18 give a complete solution of the problem of the general artificial line ; they are exactly of the same form as those for the T and Π sections, and those for the uniform transmission line.

There is, however, an ambiguity in determining θ from tanh θ and coth θ, since $\tanh \theta = \tanh (\theta + j\pi)$. This means in practice an uncertainty of 180° in phase at the output terminals. The value of θ must be chosen so that the receiving-end impedance of a single section short-circuited at the output is $Z_0 \sinh \theta$ and not $- Z_0 \sinh \theta$.

It is now possible to construct and discuss artificial lines of an unlimited number of types and of any degree of complexity. The algebra is, however, except for the simpler types, very heavy.

§ 15. The Bisection Property of a Class of Artificial Line Section.

Consider the T section shown in Fig. 28 (a) ; it may be redrawn as in Fig. 28 (b) as consisting of two equal asymmetrical networks, such as in Fig. 28 (c), connected back to back.

It will be observed that the impedance of this half-section measured at the terminals T_1T_2 is A, i.e. $Z_0 \tanh \theta/2$ if $T_1'T_2'$ are

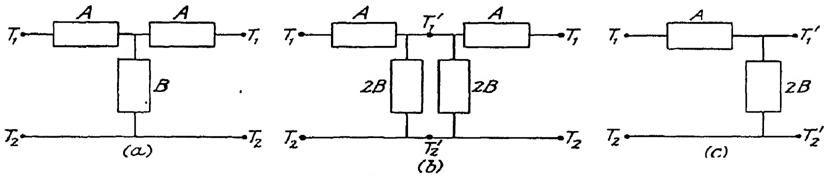


FIG. 28.

short-circuited, while it is A + 2B, i.e. $Z_0 \coth \theta/2$, if $T_1'T_2'$ are open-circuited.

It will now be shown that this is a special case of a general theorem which, where it can be applied, saves a great deal of labour in calculation.

The theorem may be stated thus : If an artificial line section

has within it n terminals T_1', T_2' . . . T_n', such that if these terminals are open, the artificial line section is cut into two exactly similar halves; the impedance of one half measured at the terminals T_1T_2 with T_1', T_2' . . . T_n' free is $Z_0 \coth \theta/2$, and with T_1', T_2' . . . T_n' all connected together is $Z_0 \tanh \theta/2$.

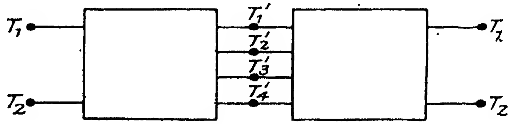


FIG. 29.

Thus, if Fig. 29 represents the artificial line section (taking $n = 4$), then $Z_0 \coth \theta/2$ is the impedance of Fig. 30 (a) and $Z_0 \tanh \theta/2$ of Fig. 30 (b).

The proof, though simple, is rather long. To simplify matters, suppose, as above, that $n = 4$, and let the artificial line section be as shown diagrammatically in Fig. 31.

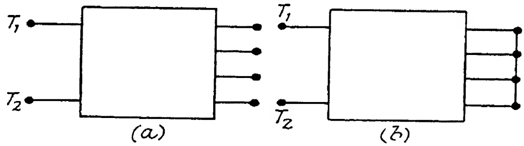


FIG. 30.

Let a_{11} be the sum of the impedances round the terminal meshes; let a_{22}, a_{33}, and a_{44} be the sums of impedances round meshes inside each half-section; $2a_{55}$, $2a_{66}$, and $2a_{77}$ be the same for the meshes common to the two half-sections. In addition, there will be the mutual impedances a_{12}, etc., between the meshes a_{11} and a_{22}, etc., in each half.

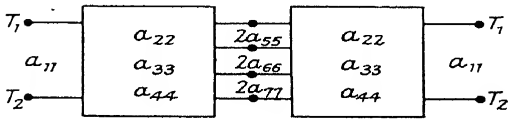


FIG. 31.

Then the characteristic impedance will be given by

$$Z_0^2 = \Delta \Big/ \frac{\partial^2 \Delta}{\partial a_{11} \partial a_{11}},$$

where Δ is the axi-symmetric determinant.

$$\begin{vmatrix}
a_{11}, & a_{12}, & a_{13}, & a_{14}, & a_{15}, & a_{16}, & a_{17}, & 0, & 0, & 0, & 0 \\
a_{21}, & a_{22}, & a_{23}, & a_{24}, & a_{25}, & a_{26}, & a_{27}, & 0, & 0, & 0, & 0 \\
a_{31}, & a_{32}, & a_{33}, & a_{34}, & a_{35}, & a_{36}, & a_{37}, & 0, & 0, & 0, & 0 \\
a_{41}, & a_{42}, & a_{43}, & a_{44}, & a_{45}, & a_{46}, & a_{47}, & 0, & 0, & 0, & 0 \\
a_{51}, & a_{52}, & a_{53}, & a_{54}, & 2a_{55}, & 2a_{56}, & 2a_{57}, & a_{54}, & a_{53}, & a_{52}, & a_{51} \\
a_{61}, & a_{62}, & a_{63}, & a_{64}, & 2a_{65}, & 2a_{66}, & 2a_{67}, & a_{64}, & a_{63}, & a_{62}, & a_{61} \\
a_{71}, & a_{72}, & a_{73}, & a_{74}, & 2a_{75}, & 2a_{76}, & 2a_{77}, & a_{74}, & a_{73}, & a_{72}, & a_{71} \\
0, & 0, & 0, & 0, & a_{45}, & a_{46}, & a_{47}, & a_{44}, & a_{43}, & a_{42}, & a_{41} \\
0, & 0, & 0, & 0, & a_{35}, & a_{36}, & a_{37}, & a_{34}, & a_{33}, & a_{32}, & a_{31} \\
0, & 0, & 0, & 0, & a_{25}, & a_{26}, & a_{27}, & a_{24}, & a_{23}, & a_{22}, & a_{21} \\
0, & 0, & 0, & 0, & a_{15}, & a_{16}, & a_{17}, & a_{14}, & a_{13}, & a_{12}, & a_{11}
\end{vmatrix},$$

in which $$a_{rs} = a_{sr}$$

This determinant, which is partially centro-symmetric, can be simplified in the following way: To the first row add the last row, to the second row add the last but one row, and similarly for the third and fourth rows. Next, in the resulting determinant: from the last column subtract the first column, from the last but one column subtract the second column, and treat similarly the last but two and last but three columns. It is then found that the determinant breaks up into the product of two other determinants, giving

$$\Delta = 2^3 \Delta' \Delta'',$$

where $\Delta' =$

$$\begin{vmatrix}
a_{11}, & a_{12}, & a_{13}, & a_{14}, & a_{15}, & a_{16}, & a_{17} \\
a_{21}, & a_{22}, & a_{23}, & a_{24}, & a_{25}, & a_{26}, & a_{27} \\
a_{31}, & a_{32}, & a_{33}, & a_{34}, & a_{35}, & a_{36}, & a_{37} \\
a_{41}, & a_{42}, & a_{43}, & a_{44}, & a_{45}, & a_{46}, & a_{47} \\
a_{51}, & a_{52}, & a_{53}, & a_{54}, & a_{55}, & a_{56}, & a_{57} \\
a_{61}, & a_{62}, & a_{63}, & a_{64}, & a_{65}, & a_{66}, & a_{67} \\
a_{71}, & a_{72}, & a_{73}, & a_{74}, & a_{75}, & a_{76}, & a_{77}
\end{vmatrix}$$

and $\Delta'' =$

$$\begin{vmatrix}
a_{11}, & a_{12}, & a_{13}, & a_{14} \\
a_{21}, & a_{22}, & a_{23}, & a_{24} \\
a_{31}, & a_{32}, & a_{33}, & a_{34} \\
a_{41}, & a_{42}, & a_{43}, & a_{44}
\end{vmatrix},$$

Similarly,

$$\frac{\partial^2 \Delta}{\partial a_{11} \partial a_{11}} = 2^3 \frac{\partial \Delta'}{\partial a_{11}} \cdot \frac{\partial \Delta''}{\partial a_{11}},$$

so that

$$Z_0^2 = \frac{\Delta'}{\dfrac{\partial \Delta'}{\partial a_{11}}} \cdot \frac{\Delta''}{\dfrac{\partial \Delta''}{\partial a_{11}}}.$$

But $\Delta' \Big/ \dfrac{\partial \Delta'}{\partial a_{11}}$ is seen to be the impedance of the first half of the network of Fig. 29 with all the terminals $T_1' T_2' T_3' T_4'$ connected together and thus is the impedance measured at $T_1 T_2$ of Fig. 30*b*, while $\Delta'' \Big/ \dfrac{\partial \Delta''}{\partial a_{11}}$ is the impedance of the same network with $T_1' T_2' T_3' T_4'$ all free and is therefore the impedance at $T_1 T_2$ of Fig. 30*a*.

Since their product is equal to Z_0^2,

$$\Delta' \Big/ \frac{\partial \Delta'}{\partial a_{11}} \quad \text{and} \quad \Delta'' \Big/ \frac{\partial \Delta''}{\partial a_{11}}$$

may be written as

$$Z_0 \tanh \psi/2 \quad \text{and} \quad Z_0 \coth \psi/2.$$

By using the identity

$$Z_0 \sinh \psi = 2 \Big/ \left(1/Z_0 \tanh \frac{\psi}{2} - 1/Z_0 \coth \frac{\psi}{2}\right),$$

and inserting values in the right-hand side, $Z_0 \sinh \psi$ can be found; if this is done, it will be found after reduction to be equal to Δ divided by the determinant obtained by omitting from Δ the first row and last column. But this is the receiving-end impedance of the original artificial line section short-circuited at the receiving-end, and is equal to $Z_0 \sinh \theta$; hence

$$\psi = \theta,$$

and the theorem is proved for this case.

The method, however, is quite general, and therefore the general theorem holds.

This result often offers an easy and rapid way of determining the constants of an artificial line.

Take the case of the three-element section of Fig. 32.

Its characteristic impedance could be worked out according to the method of § 14, but by using the above method we see that $Z_0 \tanh \theta/2$ is the impedance of the network of Fig. 33, short-circuited at $T_1' T_2'$,

i.e. $$Z_0 \tanh \frac{\theta}{2} = A + \frac{BC}{2B + C}.$$

The impedance of this network with $T_1'T_2'$ free is

$$Z_0 \coth \theta/2 = A + B.$$

Hence

$$Z_0^2 = (A + B)\left(A + \frac{BC}{2B + C}\right).$$

Many other applications of this theorem will be found in later sections.

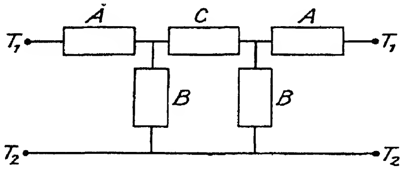


FIG. 32.

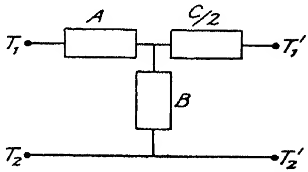


FIG. 33.

§ 16. The Two-Element Bridge Section.

The T and Π sections are both two-element sections since their structure necessitates two separate impedance elements, A and B. Another two-element section of equal and in some ways greater importance is shown in Fig. 34.

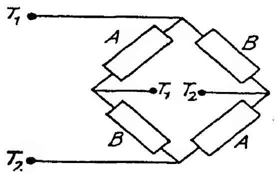


FIG. 34.

Applying the general rules,

$$Z_0 \tanh \theta = Z_c = 2\frac{1}{1/A + 1/B}$$

$$= \frac{2AB}{A + B} \quad . \quad . \quad 16.1$$

and $$Z_0 \coth \theta = \frac{A + B}{2} \quad . \quad . \quad 16.2$$

$$\therefore \quad Z_0^2 = AB \quad . \quad . \quad . \quad 16.3$$

By eliminating B between 16.1 and 16.2 A can be expressed in terms of Z_0 and θ.

Thus $$B = 2Z_0 \coth \theta - A.$$

$$\therefore \quad Z_0 \tanh \theta = \frac{2A(2Z_0 \coth \theta - A)}{2Z_0 \coth \theta},$$

$$\therefore \quad 2A^2 - 4AZ_0 \coth \theta + 2Z_0^2 = 0.$$

Hence $$A = Z_0 (\coth \theta \pm \sqrt{(\coth^2 \theta - 1)})$$

$$= Z_0 \coth \theta/2 \text{ or } Z_0 \tanh \theta/2.$$

$$\therefore \quad B = Z_0 \tanh \theta/2 \text{ or } Z_0 \coth \theta/2.$$

The ambiguity may be removed by the method given in § 14.

First try
$$A = Z_0 \coth \theta/2$$
$$B = Z_0 \tanh \theta/2.$$

Then
$$Z_0 \sinh \theta = 2\frac{1}{\dfrac{1}{Z_0 \tanh \theta/2} - \dfrac{1}{Z_0 \coth \theta/2}}$$
$$= 2\frac{1}{1/B - 1/A}$$
$$= \frac{2AB}{A - B}.$$

But if we work out the value of $Z_0 \sinh \theta$, which is the receiving-end impedance with the output terminals short-circuited, from first principles it is found to be $2AB/(B - A)$ which is of the opposite sign.

Hence we must have

$$\left.\begin{aligned} A &= Z_0 \tanh \theta/2 \\ B &= Z_0 \coth \theta/2 \end{aligned}\right\} \quad . \quad . \quad . \quad 16.4$$

The formulæ for the bridge section are thus simpler than those for the T and Π, and there is a skew symmetry that is entirely lacking in the T and Π section. This simplicity is accompanied by a greater flexibility, for it will now be shown that, corresponding to every T and Π section, an exactly equivalent bridge section can always be constructed; the converse, that a T or Π section can be constructed exactly equivalent to any bridge section does not usually hold.

For, consider a T section whose constants are Z_0' and θ', and whose elements are A' and B' (the dashes being introduced to avoid confusion).

Then since
$$Z_0' \tanh \theta'/2 = A'$$
$$Z_0' \coth \theta'/2 = A' + 2B'$$

we can construct a bridge section having A' and $A' + 2B'$ as impedance elements in the arms, as shown in Fig. 35(*b*).

Comparing this with Fig. 35(*a*) it is at once seen that this section has a characteristic impedance Z_0' and propagation constant θ', and is therefore exactly equivalent to the original T section.

Similarly, there is always a bridge exactly equivalent to a Π section, as is shown in Fig. 36.

These results can be put in a general form, for in the case of any artificial line section to which the bisection theorem may be applied $Z_0 \coth \theta/2$ and $Z_0 \tanh \theta/2$ can both be physically realised and therefore the bridge section shown in Fig. 37 is equivalent to the section of Fig. 38.

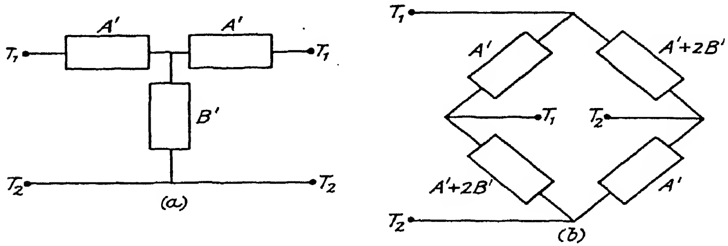


FIG. 35.

The equivalent bridges for the T and Π sections shown in Figs. 35 and 36 are simple cases of this result.

Consider now the converse, viz.: Given a bridge section of constants Z_0 and θ and arms A and B, so that $A = Z_0 \tanh \theta/2$,

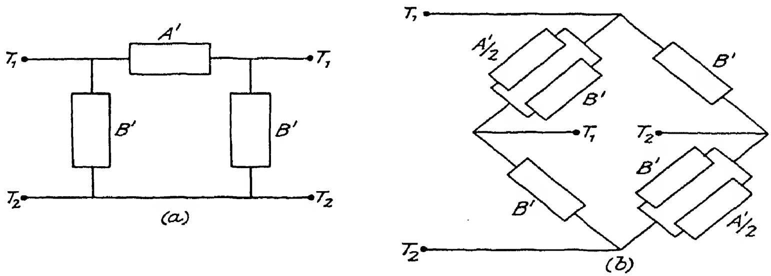


FIG. 36.

$B = Z_0 \coth \theta/2$, it is required to construct a T section exactly equivalent.

To construct the T we require for the series member the impedance $Z_0 \tanh \theta/2$. Since $A = Z_0 \tanh \theta/2$ we can make A the series member straightway. However, for the shunt impedance we have to obtain $Z_0 / \sinh \theta$.

Now $$Z_0 / \sinh \theta = \tfrac{1}{2}(Z_0 \coth \theta/2 - Z_0 \tanh \theta/2)$$
$$= B/2 - A/2,$$

and in general this impedance cannot be physically constructed without the use of negative impedances.

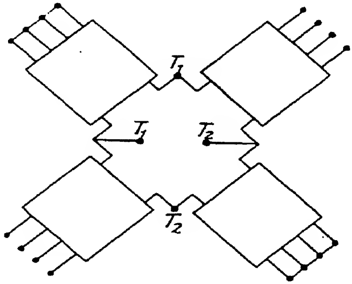


FIG. 37.

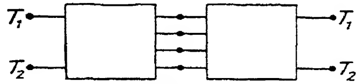


FIG. 38.

§ 17. The Equivalent Bridge Section for a Uniform and an Artificial Line.

Just as in Chapter I., § 11, the T and Π sections are equivalent to a length l of uniform line, so the bridge section of Fig. 39 is also equivalent to a length l of uniform line.

The bridge section of Fig. 40 is in the same way equivalent to n sections of an artificial line of constants Z_0 and θ.

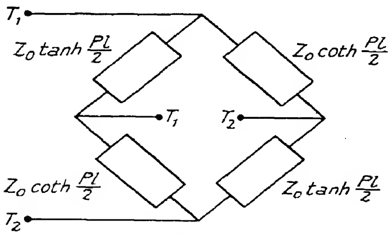


FIG. 39.

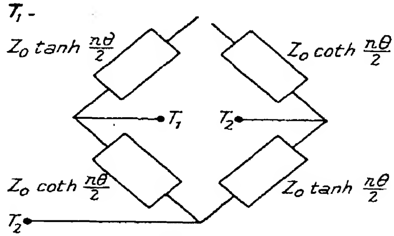


FIG. 40.

§ 18. Some Simple Three-Element Artificial Lines.

Consider the bridge network of Fig. 41 with the condition that it must be symmetrical.

It will be symmetrical ($\phi = 0$) if $Z_f = Z_f'$, i.e. if

$$\frac{(A + D)(B + C)}{A + B + C + D} = \frac{(A + B)(C + D)}{A + B + C + D},$$

i.e. $$(A - C)(B - D) = 0.$$

Thus if either of the pairs of opposite arms are equal the network is symmetrical and becomes an artificial line section. Putting $C = A$, as in Fig. 42, a simple three-element artificial line section is obtained—*The Three-Element Bridge Section.*

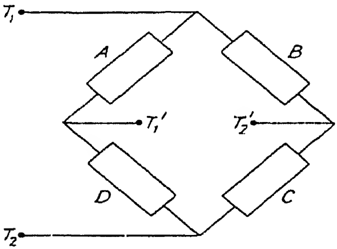


FIG. 41.

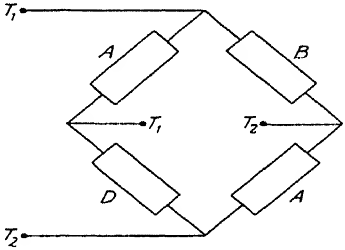


FIG. 42.

Its characteristic impedance and propagation constant are determined by

$$\left.\begin{aligned} Z_0 \tanh\theta &= Z_c = Z_c' = \frac{A(AB + 2BD + AD)}{(A + B)(A + D)} \\ Z_0 \coth\theta &= Z_f = Z_f' = \frac{(A + B)(A + D)}{2A + B + D} \\ Z_0^2 &= A \,.\, \frac{AB + AD + 2BD}{2A + B + D} \end{aligned}\right\} \quad 18.1$$

This artificial line section is of interest as it contains the ordinary Π, T and bridge section two-element artificial line sections as special cases; thus if $B = D$ it becomes the ordinary bridge section, if $D = 0$ a Π section and if $D = \infty$ a T section.

Another form of three-element artificial line is that shown in Fig. 43—*The Bridge T Section.*

By the method of § 13 the characteristic impedance and propagation constant are given by

$$\left.\begin{aligned} Z_0 \tanh \theta &= Z_c = \frac{D(A^2 + 2AB)}{A^2 + 2AB + D(A + B)} \\ Z_0 \coth \theta &= Z_f = \frac{A^2 + 2AB + D(A + B)}{2A + D} \\ Z_0^2 &= \frac{AD(A + 2B)}{2A + D} \end{aligned}\right\} \quad . \quad 18.2$$

This three-element artificial line obviously contains the T and Π sections as special cases.

The expression for Z_0 takes a simple form if $4B = D$,

when $$Z_0 = \sqrt{AD/2} = \sqrt{2AB} \quad . \quad . \quad . \quad 18.3$$

and as in the case of the bridge section is the geometric mean of two entirely independent impedances.

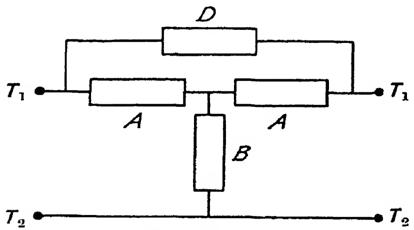


FIG. 43.

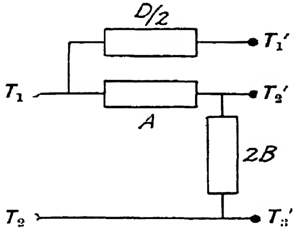


FIG. 44.

More generally, Z_0 is of the form $\sqrt{xAD}$, where x is a positive numerical quantity, if

$$A + 2B = x(2A + D),$$

i.e. $$2B = A(2x - 1) + xD.$$

Obviously any value of $x \nless \frac{1}{2}$ will give a solution that is physically possible; the case above ($D = 4B$) occurs when $x = \frac{1}{2}$.

The three-element section of Fig. 43 affords a good example of the use of the bisection property, for it may be regarded as composed of two half-sections, as in Fig. 44.

By inspection we see at once that, if T_1' T_2' T_3' are short-circuited,

$$Z_0 \tanh \theta/2 = \frac{1}{\frac{1}{A} + \frac{1}{D/2}} = \frac{AD}{2A + D}$$

and if T_1' T_2' T_3' are all free

$$Z_0 \coth \theta/2 = A + 2B,$$

whence

$$Z_0^2 = \frac{AD(A + 2B)}{2A + D}.$$

The economy of labour is obvious.

Some interesting and important applications of this artificial line will be made in Chapters III., V. and VII.

Other three-element sections that might be discussed are shown in Figs. 45 and 46. They are, however, simple cases of a general type of section which will be treated in the next chapter.

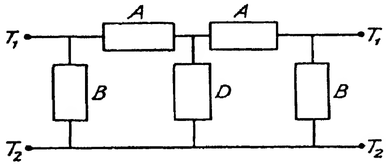


FIG. 45.

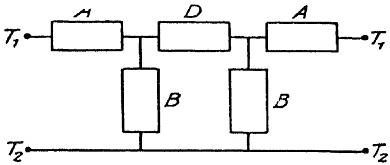


FIG. 46.

EXAMPLES AND NOTES.

1. If with three particular values z', z'', z''' of z, the terminal impedance, the corresponding values of z_1, the sending-end impedance of a network, are z_1', z_1'' and z_1''' respectively, then the homographic transformation relating z and z_1 is

$$\frac{(z_1 - z_1')(z_1'' - z_1''')}{(z_1 - z_1'')(z_1' - z_1''')} = \frac{(z - z')(z'' - z''')}{(z - z'')(z' - z''')}.$$

If this is reduced to the form

$$z_1 = \frac{az + b}{cz + d}$$

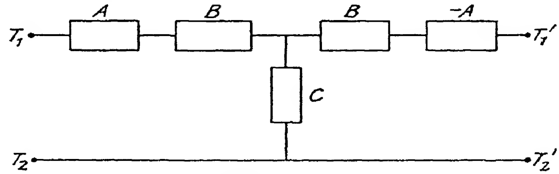
we have

$$\begin{aligned} a = {} & (z' - z'')z_1'z_1'' \\ & + (z'' - z''')z_1''z_1''' \\ & + (z''' - z')z_1'''z_1' \\ b = {} & z'z''z_1'''(z_1' - z_1'') \\ & + z''z'''z_1'(z_1'' - z_1''') \\ & + z'''z'z_1''(z_1''' - z_1') \\ c = {} & z'(z_1'' - z_1''') \\ & + z''(z_1''' - z_1') \\ & + z'''(z_1' - z_1'') \\ d = {} & z'z''(z_1' - z_1'') \\ & + z''z'''(z_1'' - z_1''') \\ & + z'''z'(z_1''' - z_1'). \end{aligned}$$

Comparing this transformation with that of equation 13.16 in which n is put equal to unity,

$$a = kZ \cosh (\theta + \phi),$$
$$b = kZ^2 \sinh \theta,$$
$$c = k \sinh \theta,$$
$$d = kZ \cosh (\theta - \phi),$$

where k is some constant.



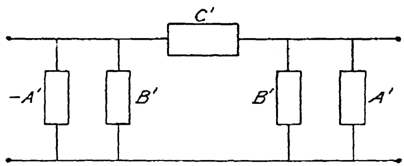
$$A = Z \sinh \phi.$$
$$B = Z \cosh \phi \tanh n\theta/2.$$
$$C = Z \cosh \phi/\sinh n\theta.$$

FIG. 47.

From these equations Z, θ and ϕ can be expressed in terms of z', z'', z''', z_1', z_1'', z_1'''.

The constants of a network can therefore be determined from three sending-end impedance measurements made at one end with three known terminating impedances at the other end.

2. The asymmetrical T network shown above in Fig. 47 is equivalent to a repeated network of n sections of the type discussed at the beginning of the chapter, and therefore the n sections of the original network are equivalent to n sections of an artificial line of which the characteristic impedance is Z $\cosh \phi$ and the propagation constant θ, preceded by a series impedance Z $\sinh \phi$, and followed by a series impedance — Z $\sinh \phi$.



$$A' = Z/\sinh \phi.$$
$$B' = Z \coth \frac{n\theta}{2}\Big/ \cosh \phi.$$
$$C' = Z \sinh n\theta/\cosh \phi.$$

FIG. 48.

3. The asymmetrical Π network shown in Fig. 48 is also equivalent to n sections of the repeated network, and therefore the n sections of the original network are also equivalent to n sections of an artificial line of characteristic impedance Z/$\cosh \phi$ and propagation constant θ, preceded by a shunt impedance — Z/$\sinh \phi$, and followed by a shunt impedance Z/$\sinh \phi$.

The network of Fig. 48 can be derived from Fig. 47 by means of the Star-Delta transformation.

4. Prove that in the asymmetrical network shown in Fig. 49

$$\phi = \theta/2.$$

5. If a network is constructed of n_1 artificial line sections having a characteristic impedance Z_0 and propagation constant θ_1, followed by n_2 sections of a different artificial line having the same Z_0 but a propagation constant θ_2, then the network is symmetrical and may be regarded as a single artificial line section of characteristic impedance Z_0 and propagation constant

$$n_1\theta_1 + n_2\theta_2.$$

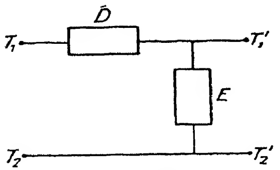


FIG. 49.

6. Properties of Repeated Networks.

(*a*) The input impedance of the general repeated network of n sections with T_1T_2 as input will be independent of z if

$$n\theta + \phi = j(\pi/2 + r\pi),$$

where r is any integer.

(*b*) The voltage across z will be independent of the value of z if

$$n\theta = jr\pi,$$

where r is any integer.

(*c*) The current through z will be zero if

$$\phi = j\frac{\pi}{2}.$$

7. For a given type of artificial line section composed entirely of resistances, show that, if for a given frequency y_1, y_2, y_3, etc., are the receiving-end impedances of 1, 2, 3, etc., sections terminated by $x' + jy'$, then y_1, y_2, y_3, etc., lie on the hyperbola

$$\left(\frac{x}{y'}\right)^2 - \frac{1}{Z_0^2}\left(x - \frac{x'y}{y'}\right)^2 = 1.$$

8. If, in § 13, $\phi = 0$ then

$$\xi = -\zeta = Z_0.$$

CHAPTER III.

LADDER NETWORKS AND GENERALISATIONS OF THE T AND Π SECTION ARTIFICIAL LINES.

§ 19. Continued Fractions.

THE T and Π artificial lines dealt with in Chapter I. were derived from a simple ladder network having all the series and all the shunt elements equal; in this chapter the theory of a general ladder network, in which all the elements may have arbitrary values, will first be considered, and then from it will be derived a class of symmetrical ladder artificial lines, of which the T and Π section lines are but simple cases. As a preliminary it will be necessary to enter, in some detail, into the theory of continued fractions and the allied functions known as simple continuants.

The type of continued fraction that has to be considered is of the form :

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \ddots}}}$$

or in a more compact form with dropped +'s :

$$a_1 + \frac{1}{a_2 +}\,\frac{1}{a_3 +}\,\frac{1}{a_4 +}\,\cdots$$

having all numerators unity. The brief treatment given below will be based on Chrystal's "Text-Book of Algebra," Chapter XXXIV., and Muir's "Theory of Determinants," 1882 (Macmillan), Chapter III., which may be consulted, together with Muir's larger work on determinants for a more extended treatment.

The fraction obtained from the above continued fraction by retaining only the first n of the a's is called the nth convergent, and

when reduced to a simple fraction by direct simplification, without any cancellation of possible common factors, it is denoted by

$$\frac{p_n}{q_n}.$$

Thus the first convergent is simply a_1. In this case

$$p_1 = a,$$
$$q_1 = 1.$$

The second convergent,

$$\frac{p_2}{q_2} = a_1 + \frac{1}{a_2} = \frac{a_1a_2 + 1}{a_2},$$

i.e.

$$p_2 = a_1a_2 + 1,$$
$$q_2 = a_2.$$

Similarly, for the third and fourth convergents,

$$\frac{p_3}{q_3} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3}} = \frac{a_1a_2a_3 + a_1 + a_3}{a_2a_3 + 1}$$

$$\frac{p_4}{q_4} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4}}} = \frac{a_1a_2a_3a_4 + a_1a_2 + a_1a_4 + a_3a_4 + 1}{a_2a_3a_4 + a_2 + a_4}.$$

If the continued fraction terminates after n a's, then, of course, the nth convergent is the value of the continued fraction itself; if, however, the continued fraction is infinite the successive convergents will, in general, (provided that the a's have their real parts positive and not zero) be successive approximations to the continued fraction.

Successive p's and q's are related by simple recursion formulæ, for consider the two tables of p's and q's:

$$p_1 = a_1.$$
$$p_2 = a_1a_2 + 1.$$
$$p_3 = a_1a_2a_3 + a_1 + a_3.$$
$$p_4 = a_1a_2a_3a_4 + a_1a_2 + a_1a_4 + a_3a_4 + 1.$$
$$q_1 = 1.$$
$$q_2 = a_2.$$
$$q_3 = a_2a_3 + 1.$$
$$q_4 = a_2a_3a_4 + a_2 + a_4.$$

It is at once seen that

$$\begin{aligned} p_3 &= a_3 p_2 + p_1. \\ p_4 &= a_4 p_3 + p_2. \\ q_3 &= a_3 q_2 + q_1. \\ q_4 &= a_4 q_3 + q_2. \end{aligned}$$

This suggests that in general

$$\begin{aligned} p_n &= a_n p_{n-1} + p_{n-2}. \\ q_n &= a_n q_{n-1} + q_{n-2}. \end{aligned}$$

To prove this assume that these relations are true for the ($n - 1$)th convergent so that

$$\frac{p_{n-1}}{q_{n-1}} = \frac{a_{n-1} p_{n-2} + p_{n-3}}{a_{n-1} q_{n-2} + q_{n-3}}.$$

Now consider the nth convergent:

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \ddots \cfrac{}{\cfrac{1}{a_{n-2} + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}}}}}.$$

It is clear that the nth convergent is obtained from the ($n - 1$)th by replacing

$$a_{n-1} \text{ by } a_{n-1} + \frac{1}{a_n}.$$

Hence, making this substitution,

$$\begin{aligned} \frac{p_n}{q_n} &= \frac{\left(a_{n-1} + \frac{1}{a_n}\right) p_{n-2} + p_{n-3}}{\left(a_{n-1} + \frac{1}{a_n}\right) q_{n-2} + q_{n-3}} \\ &= \frac{a_n(a_{n-1} p_{n-2} + p_{n-3}) + p_{n-2}}{a_n(a_{n-1} q_{n-2} + q_{n-3}) + q_{n-2}} \\ &= \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}}. \end{aligned}$$

Hence if the assumption is true for the ($n - 1$)th convergent it is true for the nth convergent. But the relations hold for the

third and fourth; therefore by the above they hold for the fifth, and so on for all higher values.

By the use of these rules the numerators and denominators of successive convergents may be built up to any extent.

§ 20. Continuants.

The theory of continued fractions is greatly simplified by means of certain functions called *Continuants*, originally due to Euler.

It will be seen that, apart from its specific relation to the continued fraction, p_n is a rational integral function of the n quantities $a_1, a_2, \ldots a_n$ determined by a set of equations

$$\begin{aligned} p_2 &= a_2 p_1 + p_0, \\ p_3 &= a_3 p_2 + p_1, \\ p_4 &= a_4 p_3 + p_2, \\ &\text{etc., etc.,} \end{aligned}$$

together with

$$\begin{aligned} p_1 &= a_1, \\ p_0 &= 0. \end{aligned}$$

The function, p_n, of $a_1, a_2, \ldots a_n$, defined in this way, is termed a *Simple Continuant of the nth order* of $a_1, a_2, \ldots a_n$, and is denoted by

$$\mathrm{K}(a_1, a_2, \ldots a_n).$$

Thus simple continuants are defined by the set of relations

$$\begin{aligned} \mathrm{K}(0) &= 1 \\ \mathrm{K}(a_1) &= a_1 \\ \mathrm{K}(a_1, a_2) &= a_2\mathrm{K}(a_1) + \mathrm{K}(0) \\ \mathrm{K}(a_1, a_2, a_3) &= a_3\mathrm{K}(a_1, a_2) + \mathrm{K}(a_1) \\ \mathrm{K}(a_1, a_2, a_3, a_4) &= a_4\mathrm{K}(a_1, a_2, a_3) + \mathrm{K}(a_1, a_2) \\ &\cdots\cdots\cdots \end{aligned}$$

$$\mathrm{K}(a_1, a_2, \ldots a_n) = a_n\mathrm{K}(a_1, a_2, \ldots a_{n-1}) + \mathrm{K}(a_1, a_2 \ldots a_{n-2}),$$

a definition complete in itself from which all reference to continued fractions has been removed.

It is important to distinguish between the value of the zero order continuant, K(0), i.e. unity, and the value of the first order continuant $\mathrm{K}(a_1)$ when $a_1 = 0$, which is zero. The latter must be written

$$\underset{a_1 = 0}{\mathrm{K}(a_1)} = 0.$$

Now consider the q's; q_2 is the same function of a_2 as p_1 is of a_1; q_3 is the same function of a_2, a_3 as p_2 is of a_1, a_2. In general,

q_n is the same function of $a_2, a_3, \ldots a_n$ as p_{n-1} is of $a_1, a_2, \ldots a_{n-1}$. Hence the q's are also continuants and

$$q_n = \mathrm{K}(a_2, a_3, \ldots a_n).$$

In particular,

$$q_2 = \mathrm{K}(a_2) = a_2,$$
$$q_1 = \mathrm{K}(0) = 1.$$

Hence the successive convergents to the continued fraction are

$$a_1 = \frac{p_1}{q_1} = \frac{\mathrm{K}(a_1)}{\mathrm{K}(0)}$$

$$a_1 + \frac{1}{a_2} = \frac{p_2}{q_2} = \frac{\mathrm{K}(a_1, a_2)}{\mathrm{K}(a_2)}$$

$$\cdots\cdots\cdots$$

$$a_1 + \frac{1}{a_2 +} \frac{1}{a_3 +} \ldots \frac{1}{a_n} = \frac{p_n}{q_n} = \frac{\mathrm{K}(a_1, a_2, \ldots a_n)}{\mathrm{K}(a_2, a_3, \ldots a_n)}.$$

Expansion of Simple Continuants (Hindenburg's Rule).

The expansion of a continuant of fairly high order by means of the fundamental definitions may be a long process. The work may, however, be simplified by using the following schematic arrangement due to Hindenburg (1741-1808). Suppose it is required to develop the series of simple continuants $\mathrm{K}(a_1)$, $\mathrm{K}(a_1, a_2)$, etc., then construct a table as in Fig. 50.

Begin by writing down a_1 and enclosing it in a rectangle 1, 1. This gives $\mathrm{K}(a_1) = a_1$.

Secondly, write a_2 to the right of a_1, and write 1 below a_1; enclose in a rectangle 2, 2. Then the rows in 2, 2 give $\mathrm{K}(a_1, a_2)$, i.e. $a_1a_2 + 1$.

Next write a_3 at the ends of all the rows in 2, 2 and repeat under the rectangle, 2, 2 all the rows in 1, 1, and finally put unity in the blank space of the column of a_3's. Enclose in the rectangle 3, 3. Then the rows of 3, 3 give $\mathrm{K}(a_1, a_2, a_3)$.

To obtain higher order continuants the scheme is continued in a similar manner which will be obvious from the figure.

This scheme leads to a simple rule, due to Euler, for writing down the expansion of a continuant of the nth order.

Write down $a_1a_2a_3 \ldots a_n$. This is the first term. To find the remaining terms omit from this product in all possible ways one or more pairs of consecutive a's, always replacing each consecutive pair of a's omitted by unity.

For example, to expand $\mathrm{K}(a_1, a_2)$ write down the first term a_1a_2;

from this the only way that consecutive pairs can be omitted is to omit a_1a_2; replacing it by unity gives $K(a_1, a_2) = a_1a_2 + 1$.

Again, to expand $K(a_1, a_2, a_3)$ write down the first term $a_1a_2a_3$; the consecutive pairs a_1a_2 and a_2a_3 can be omitted, giving the terms a_3 and a_1. It is impossible to omit two pairs of a's, so the process stops and therefore $K(a_1, a_2, a_3) = a_1a_2a_3 + a_1 + a_3$.

In expanding a continuant of the nth order by Euler's Rule the first term stands alone; the number of terms obtained by the process of omitting one pair of a's is $n - 1$ and generally the total number of terms obtained by omitting r pairs of a's is given by

	1	2	3	4	5	6
1	a_1	a_2	a_3	a_4	a_5	a_6
2	1		a_3	a_4	a_5	a_6
3	a_1		1	a_4	a_5	a_6
	a_1	a_2		1	a_5	a_6
4	1			1	a_5	a_6
	a_1	a_2	a_3		1	a_6
	1		a_3		1	a_6
5	a_1		1		1	a_6
	a_1	a_2	a_3	a_4		1
	1		a_3	a_4		1
	a_1		1	a_4		1
	a_1	a_2		1		1
6	1			1		1

FIG. 50.

$$\frac{(n - r)(n - r - 1) \; . \; . \; . \; (n - 2r + 1)}{r\,!}.$$

Thus the total number of terms in a continuant of the nth order is

$$1 + (n - 1) + \frac{(n - 2)(n - 3)}{2\,!} + \frac{(n - 3)(n - 4)(n - 5)}{3\,!} + \; . \; . \; .$$

which can be proved equal to

$$\frac{(1 + \sqrt{5})^{n+1} - (1 - \sqrt{5})^{n+1}}{2^{n+1}\sqrt{5}}.$$

These results furnish useful checks on expansions.

§ 21. Simple Continuants expressed as Determinants.

It was discovered by Sylvester that simple continuants can be expressed as a remarkable form of skew determinant:

$$\mathrm{K}(a_1, a_2, \ldots = \left| \begin{array}{cccccc} a_1 & 1 & 0 & 0 & . & \\ -1 & a_2 & 1 & 0 & . & \\ 0 & -1 & a_3 & 1 & . & \\ 0 & 0 & -1 & a_4 & 1 & \\ 0 & 0 & 0 & -1 & a_5 & \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{array} \right.$$

The determinant form does not actually occur in these pages, but it is of importance, for it is from this point of view that the theory has been chiefly developed. A good account of continuants from this point of view is given in Muir's "Theory of Determinants," Macmillan, 1882.

§ 22. Properties of Continuants.

Use will be made of the following theorems:—

22.1 $\mathrm{K}(a_1, a_2, \ldots, a_{n-1}, a_n) = \mathrm{K}(a_n, a_{n-1}, \ldots, a_2, a_1)$.

22.2 $\mathrm{K}(a_1, \ldots, a_n) = a_1\mathrm{K}(a_2, \ldots, a_n) + \mathrm{K}(a_3, \ldots, a_n)$.

22.3 $\mathrm{K}(a_1, \ldots, a_n) = \mathrm{K}(a_1, \ldots, a_p)\mathrm{K}(a_{p+1}, \ldots, a_n)$
$+ \mathrm{K}(a_1, \ldots, a_{p-1})\mathrm{K}(a_{p+2}, \ldots, a_n)$.

22.4 $\mathrm{K}(a_1, \ldots, a_n)\mathrm{K}(a_2, \ldots, a_{n-1})$
$= \mathrm{K}(a_1, \ldots, a_{n-1})\mathrm{K}(a_2, \ldots, a_n) + (-1)^n$.

22.5 $\mathrm{K}(0, a_2, a_3, \ldots, a_n) = \mathrm{K}(a_3, \ldots, a_n)$.

22.6 $\mathrm{K}(\ldots, a, b, c, 0, e, f, g, \ldots)$
$= \mathrm{K}(\ldots, a, b, c + e, f, g, \ldots)$.

22.7 $\mathrm{K}(\ldots, a, b, 0, 0, e, f, \ldots) = \mathrm{K}(\ldots, a, b, e, f, \ldots)$.

22.8 $\mathrm{K}(\ldots, a, b, c, 0, 0, 0, e, f, g, \ldots)$
$= \mathrm{K}(\ldots, a, b, c + e, f, g, \ldots)$.

22.9 $\mathrm{K}(a_1x^{-1}, a_2x, a_3x^{-1}, a_4x, \ldots, a_nx^{(-1)^n})$
$= \mathrm{K}(a_1, a_2, a_3, \ldots, a_n)$ if n is even,
$= \mathrm{K}(a_1, a_2, a_3, \ldots, a_n) \times x^{-1}$ if n is odd.

22.10 $\mathrm{K}(a_1, a_2, \ldots, a_{n-1}, a_n, a_{n-1}, \ldots, a_2, a_1)$
$= \mathrm{K}(a_1, a_2, \ldots, a_{n-1})\{\mathrm{K}(a_1, a_2, \ldots, a_{n-2})$
$+ \mathrm{K}(a_1, a_2, \ldots, a_n)\}$.

22.11 $K(a_1, a_2, \ldots, a_{n-1}, a_n, a_{n-1}, \ldots, a_2, a_1)$
$$= 2K(a_1, a_2, \ldots, a_{n-1})K(a_1, a_2, \ldots, a_{n-1}, a_n/2).$$

22.12 $a_n K(a_1, a_2, \ldots, a_{n-1}, a_n, a_{n-1}, \ldots, a_2, a_1)$
$$= K(a_1, a_2, \ldots, a_n)^2 - K(a_1, a_2, \ldots, a_{n-2})^2.$$

§ 23. Theory of the Ladder Network.

The distribution of current and voltage throughout a ladder network can be put into a very simple and compact form by means of continuant notation.

Let Fig. 51 represent a ladder network constructed of n impedances $x_1, x_2, x_3, \ldots x_n$, and in order to simplify the results let the impedances of the elements x_1, x_3, x_5, etc., be a_1, a_3, a_5, etc., and the impedances of the shunt elements x_2, x_4, etc., be $\frac{1}{a_2}, \frac{1}{a_4}$, etc.

The impedance of this network at T_1T_2 (i.e. the sending-end

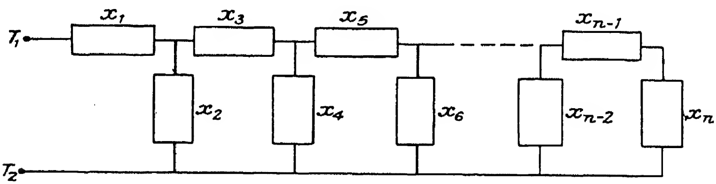


FIG. 51.

impedance) can be written down by the ordinary rules for combining series and shunt impedances and will be the continued fraction

$$x_1 + \cfrac{1}{\cfrac{1}{x_2} + \cfrac{1}{x_2 + \cfrac{1}{\cfrac{1}{x_4} + \cfrac{1}{x_5 + \cfrac{1}{\cfrac{1}{x_6} + \ldots}}}}}$$

i.e.

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{\ddots}{+ \cfrac{1}{a_n}}}}}.$$

which is equal to

$$\frac{K(a_1, a_2 \ . \ . \ . \ a_n)}{K(a_2 \ . \ . \ . \ a_n)} \qquad . \quad . \quad . \qquad 23.1$$

Consider now the current in the element x_3; current entering the network will divide between x_2 and x_3, the fraction of the current entering x_3 will be given by

$$\frac{\text{Admittance of network } x_3, x_4, x_5, \ . \ . \ . \ x_n}{\text{Admittance of } x_2 + \text{Admittance of network } x_3, x_4, x_5, \ . \ . \ . \ x_n}$$

$$= \frac{\dfrac{K(a_4 \ . \ . \ . \ a_n)}{K(a_3 \ . \ . \ . \ a_n)}}{a_2 + \dfrac{K(a_4 \ . \ . \ . \ a_n)}{K(a_3 \ . \ . \ . \ a_n)}}$$

$$= \frac{K(a_4 \ . \ . \ . \ a_n)}{a_2 K(a_3 \ . \ . \ . \ a_n) + K(a_4 \ . \ . \ . \ a_n)}$$

$$= \frac{K(a_4 \ . \ . \ . \ a_n)}{K(a_2 \ . \ . \ . \ a_n)}.$$

Similarly of this current a fraction

$$\frac{K(a_6 \ . \ . \ . \ a_n)}{K(a_4 \ . \ . \ . \ a_n)}$$

will enter x_5.

So that the fraction of the original current passing through x_5 will be

$$\frac{K(a_4 \ . \ . \ . \ a_n)}{K(a_2 \ . \ . \ . \ a_n)} \times \frac{K(a_6 \ . \ . \ . \ a_n)}{K(a_4 \ . \ . \ . \ a_n)}$$

$$= \frac{K(a_6 \ . \ . \ . \ a_n)}{K(a_2 \ . \ . \ . \ a_n)}.$$

The process can be continued and the general result is that the current through any series member x_{2r+1} is equal to the input current multiplied by

$$\frac{K(a_{2r+2} \ . \ . \ . \ a_n)}{K(a_2 \ . \ . \ . \ a_n)} \qquad . \quad . \quad . \qquad 23.2$$

and that the ratio of the current through x_{2r+1} to the current through x_{2s+1}

$$= \frac{K(a_{2r+2} \ . \ . \ . \ a_n)}{K(a_{2s+2} \ . \ . \ . \ a_n)} \qquad . \quad . \quad . \qquad 23.3$$

Let the current in the series member x_{2r+1} due to a voltage v impressed at the sending end T_1T_2 be v/Z_{2r+1} so that Z_{2r+1} can

be termed the impedance of x_{2r+1} with respect to the terminals T_1T_2; Z_{2r+1} can now be determined; the current through x_{2r+1} is $K(a_{2r+2} \ldots a_n)/K(a_2 \ldots a_n)$ times the current through x_1 and the current through x_1 is

$$v \Big/ \frac{K(a_1 \ldots a_n)}{K(a_2 \ldots a_n)},$$

and therefore the current through x_{2r+1} is equal to

$$v \Big/ \frac{K(a_1 \ldots a_n)}{K(a_{2r+2} \ldots a_n)},$$

whence

$$Z_{2r+1} = \frac{K(a_1 \ldots a_n)}{K(a_{2r+2} \ldots a_n)}. \qquad 23.4$$

Next consider the voltages across shunt members; if a voltage is applied at T_1T_2 the fraction of the voltage across x_2 will be given by

$$\frac{\text{Impedance of network } x_2, x_3, x_4, \ldots, x_n}{\text{Impedance of network } x_1, x_2, x_3, \ldots, x_n}$$

$$= \frac{K(a_3 \ldots a_n)}{K(a_2 \ldots a_n)} \Big/ \frac{K(a_1 \ldots a_n)}{K(a_2 \ldots a_n)}$$

$$= \frac{K(a_3 \ldots a_n)}{K(a_1 \ldots a_n)}.$$

Of this voltage across x_2, which is the input voltage to the network beginning at x_3, a fraction $K(a_5 \ldots a_n)/K(a_3 \ldots a_n)$ will be the voltage across x_4.

So that, of the original applied voltage, a fraction

$$\frac{K(a_5 \ldots a_n)}{K(a_1 \ldots a_n)}$$

will be the voltage across x_4. Continuing in the same way, the voltage across the shunt element x_{2r} is equal to the input voltage multiplied by

$$\frac{K(a_{2r+1} \ldots a_n)}{K(a_1 \ldots a_n)} \qquad 23.5$$

and the ratio of the voltage across x_{2r} to the voltage across x_{2s} is

$$\frac{K(a_{2r+1} \ldots a_n)}{K(a_{2s+1} \ldots a_n)}. \qquad 23.6$$

If Z_{2r} is the impedance of the shunt element x_{2r} with respect to the terminals T_1T_2, it follows that

$$Z_{2r} = \frac{1}{a_{2r}} \frac{K(a_1 \ldots a_n)}{K(a_{2r+1} \ldots a_n)}. \qquad 23.7$$

If in Fig. 51 we take x_n as a terminal impedance, then Z_n is the receiving-end impedance of the network $x_1, x_2, \ldots, x_{n-1}$ terminated by x_n; its value is given by

$$\frac{K(a_1, a_2, \ldots, a_n)}{a_n}, \quad . \quad . \quad . \quad 23.8$$

which is equal to

$$K(a_1 \ldots a_{n-1}) + \frac{K(a_1 \ldots a_{n-2})}{a_n}.$$

In particular, if x_n is of zero impedance,

$$a_n = \frac{1}{x_n} = \infty,$$

and the receiving end impedance of the network $x_1 \ldots x_{n-1}$, short-circuited at the output, is

$$K(a_1 \ldots a_{n-1}). \quad . \quad . \quad . \quad 23.9$$

It has been taken throughout that the network commences with a series member; there is no loss of generality in this, for if the network commences with a shunt member it is merely necessary to put $a_1 = 0$, and notice that

$$K(0, a_2, a_3, \ldots, a_n) = K(a_3 \ldots a_n). \quad .$$

It would be possible to proceed from this point and develop the complete theory of T and Π sections entirely afresh by evaluating continuants of the form

$$K(a, b, 2a, b, 2a, \ldots, 2a, b, a + z),$$
$$K(b, 2a, b, 2a, \ldots, 2a, b, a + z).$$

Difference equations are obtained the solution of which can be obtained by a method given in Herschel's "Finite Differences," Cambridge, 1820, Section XII.

Thus, together with the methods given in Chapters I. and II., three independent methods of deriving the properties of the T and Π section artificial lines are available.

§ 24. The Generalised Ladder Artificial Line Section.

Consider an artificial line section such as is shown in Fig. 52.

It consists of $2n - 1$ elements symmetrically placed about a centre element x_n, which may be either series or shunt.

Let the impedances of the series elements x_1, x_3, etc., be a_1, a_3, etc., and of the shunt elements x_2, x_4, etc., be $\frac{1}{a_2}, \frac{1}{a_4}$, etc.

The constants can be calculated by the standard method of Chapter II.

Thus $Z_0 \tanh \theta$ is the impedance of a single section short-circuited at the output and is therefore given by

$$Z_0 \tanh \theta = a_1 + \frac{1}{a_2 +}\,\frac{1}{a_3 +}\,\cdots\,\frac{1}{+\,a_n +}\,\cdots\,\frac{1}{+\,a_2 +}\,\frac{1}{a_1}$$

$$= \frac{K(a_1, a_2, \ldots, a_n, \ldots, a_2, a_1)}{K(a_2, a_3, \ldots, a_n, \ldots, a_2, a_1)} \quad . \quad . \quad 24.1$$

and

$$Z_0 \coth \theta = \frac{K(a_1, a_2, \ldots, a_n, \ldots, a_3, a_2)}{K(a_2, a_3, \ldots, a_n, \ldots, a_3, a_2)} . \quad . \quad . \quad 24.2$$

Thus $$Z_0^2 = \frac{K(a_1, a_2, \ldots, a_n, \ldots, a_2, a_1)}{K(a_2, a_3, \ldots, a_n, \ldots, a_3, a_2)} . \quad . \quad 24.3$$

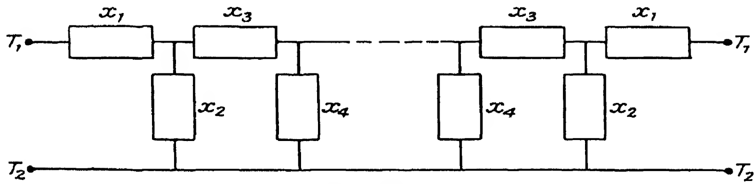


FIG. 52.

Alternatively, we can use the Bisection Theorem; there will be two cases to consider depending on whether x_n is a shunt or a series member.

A half section is shown in Fig. 53 (a) and (b), (a) showing the case where n is even and (b) showing the case where n is odd.

$Z_0 \tanh \theta/2$ is the impedance of a half section short-circuited at the output and is given by

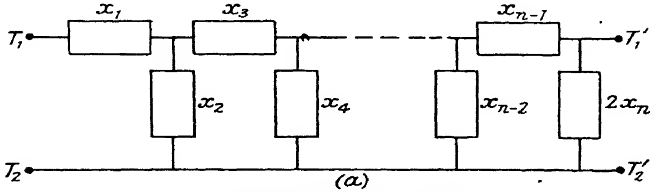
$$\left.\begin{aligned} Z_0 \tanh \theta/2 &= \frac{K(a_1, a_2, \ldots, a_{n-2}, a_{n-1})}{K(a_2, a_3, \ldots, a_{n-2}, a_{n-1})} \quad \text{if } n \text{ is even} \\ &= \frac{K(a_1, a_2, \ldots, a_{n-1}, a_n/2)}{K(a_2, a_3, \ldots, a_{n-1}, a_n/2)} \quad \text{if } n \text{ is odd} \end{aligned}\right\} \quad 24.4$$

Similarly

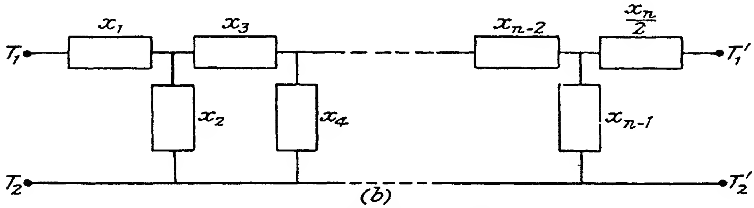
$$\left.\begin{aligned} Z_0 \coth \theta/2 &= \frac{K(a_1, a_2, \ldots, a_{n-1}, a_n/2)}{K(a_2, a_3, \ldots, a_{n-1}, a_n/2)} \quad \text{if } n \text{ is even} \\ &= \frac{K(a_1, a_2, \ldots, a_{n-2}, a_{n-1})}{K(a_2, a_3, \ldots, a_{n-2}, a_{n-1})} \quad \text{if } n \text{ is odd} \end{aligned}\right\} \quad 24{\cdot}5$$

The value of Z_0^2 obtained by multiplying $Z_0 \coth \theta/2$ by $Z_0 \tanh \theta/2$, will be found, on applying 22.11, to be the same as 24.3.

The standard equations for the T and Π sections can be derived as an exercise.



(*a*) **Half section, *n* even.**



(*b*) **Half section, *n* odd.**

Fig. 53.

§ 25. Reciprocal Impedances and Networks.

A very important and useful method of transforming ladder networks can now be dealt with.

Suppose P and Q are any two impedances, and let $S = P^2/Q$; then S will be of the dimensions of an impedance, and will be termed the reciprocal of Q with respect to P. Also since $Q = P^2/S$, Q is the reciprocal of S with respect to P. Whatever the form of P and Q at any one frequency, S could be represented by a resistance in series with either a capacity or an inductance; in general, however, the values of the resistance and the capacity or inductance would change with frequency, while the reactance might be a capacity at one frequency and an inductance at another.

There is, however, a large class of networks where the impedance S can be represented at all frequencies by a definite physical network. Consider the simple case where P is a resistance R, and Q an inductance L.

Then
$$S = \frac{P^2}{Q} = \frac{R^2}{j\omega L} = \frac{1}{j\omega\frac{L}{R^2}}.$$

But at all frequencies $\dfrac{1}{j\omega\frac{L}{R^2}}$ is the impedance of a condenser having a capacity L/R^2. Thus the reciprocal of an inductance L with respect to a resistance R is a capacity L/R^2.

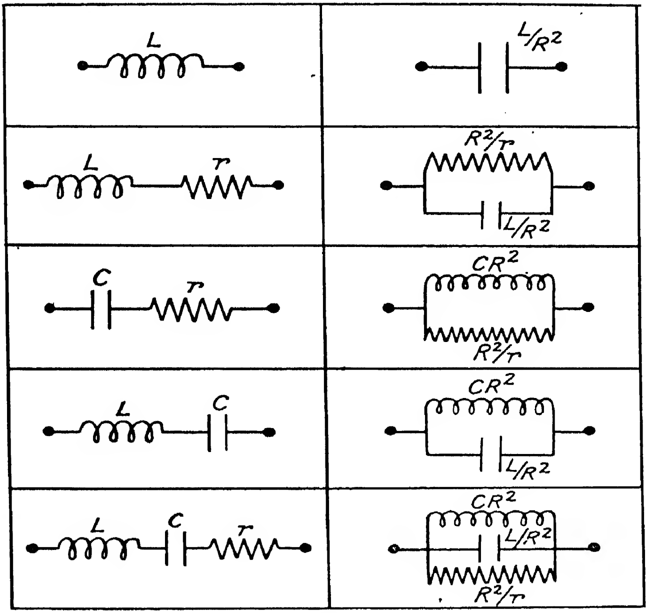


FIG. 54.

As a rather more complicated example, suppose there is a resistance r in series with L; then the reciprocal with respect to R is

$$S = \frac{R^2}{r + j\omega L} = \frac{1}{\frac{r}{R^2} + j\omega\frac{L}{R^2}}.$$

This is the impedance of a capacity, L/R^2, shunted by a resistance R^2/r.

In the last example the reciprocal is derived from the original network by replacing each element by its reciprocal and changing the series connection to a parallel one.

In Fig. 54 are shown some simple pairs of reciprocal networks, the reciprocation being with respect to a resistance R.

§ 26. A General Theorem on Reciprocal Networks.

The examples on reciprocation of the last section are simple cases of a general theorem.

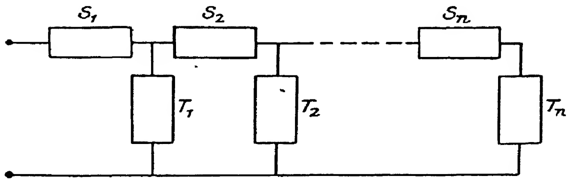


FIG. 55.

Suppose that a ladder network is given, consisting of series impedances S_1, S_2, S_3, . . ., S_n, and shunt impedances T_1, T_2, T_3, . . ., T_n, as shown in Fig. 55, and that X is its impedance;

then
$$X = \frac{K(S_1,\ 1/T_1,\ S_2,\ 1/T_2,\ \ldots,\ S_n,\ 1/T_n)}{K(1/T_1,\ S_2,\ 1/T_2,\ \ldots,\ S_n,\ 1/T_n)}.$$

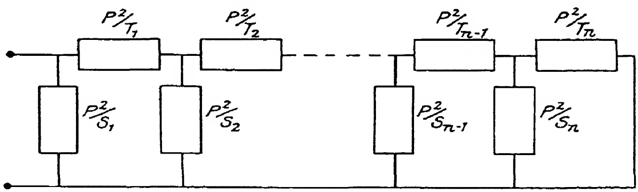


FIG. 56.

Construct a network as shown in Fig. 56 having for its series elements the reciprocals of T_1, T_2, . . ., T_n, with respect to some impedance P, i.e. P^2/T_1, P^2/T_2, . . ., P^2/T_n, and for shunt elements the reciprocals of S_1, S_2, . . ., S_n, i.e. P^2/S_1, P^2/S_2, . . ., P^2/S_n, and let the impedance of this network be Y.

Then
$$Y = \frac{K(P^2/T_1,\ S_2/P^2,\ P^2/T_2,\ S_3/P^2,\ \ldots,\ P^2/T_n)}{K(S_1/P^2,\ P^2/T_1,\ S_2/P^2,\ P^2/T_2,\ \ldots,\ P^2/T_n)}.$$

Applying 22.9 to both numerator and denominator we obtain

$$Y = \frac{P^2K(1/T_1, S_2, 1/T_2, \ldots, 1/T_n)}{K(S_1, 1/T_1, S_2, \ldots, 1/T_n)}.$$

$$= \frac{P^2}{X}.$$

Therefore the network of Fig. 56 is the reciprocal with respect to P of the network of Fig. 55, and vice versa. The process of

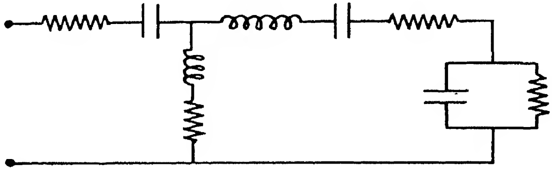


FIG. 57.

deriving the network Y from X will be spoken of as reciprocating X with respect to P.

A special case of importance is when all the shunt impedances in Fig. 55 are infinite, so that the network reduces to the n impedances $S_1, S_2, S_3, \ldots S_n$ connected in series. In the reciprocal network all the series impedances P^2/T_1, P^2/T_2, P^2/T_3, $\ldots$ P^2/T_n become zero, so that the reciprocal network consists in the n impedances P^2/S_1, P^2/S_2, $\ldots$ P^2/S_n connected in parallel.

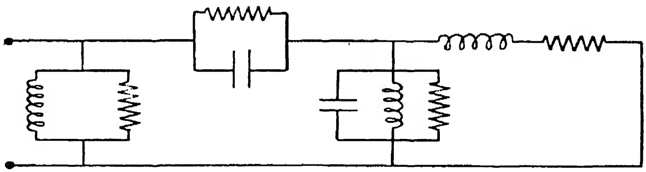


FIG. 58.

As an example take the network shown in Fig 57.

Its reciprocal with respect to a resistance R is of the form shown in Fig. 58.

It follows that a network of ladder form, if all its series and shunt elements can be reduced to simple ladder types, has a reciprocal with respect to any resistance, R, which can be physically constructed to be true for all frequencies.

Reciprocation with respect to impedances other than resistances, such as inductances or capacities, is not usually possible; for consider the reciprocals of an inductance L_1, a capacity C and a resistance R with respect to an inductance L. The reciprocals will be respectively

$$\frac{(j\omega L)^2}{j\omega L_1} = j\omega L \cdot \frac{L}{L_1}, \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{(j\omega L)}{\left(\frac{1}{j\omega C}\right)} = -j\omega^3 L^2 C, \quad . \quad . \quad . \quad (2)$$

$$\frac{(j\omega L)^2}{R} = -\frac{\omega^2 L^2}{r}. \quad . \quad . \quad . \quad . \quad (3)$$

(1) represents the impedance of an inductance $L \cdot \frac{L}{L_1}$, but neither (2) nor (3) represent the impedance of any network that can be simply constructed.

It is possible in many cases to reciprocate special networks related to artificial lines with respect to the characteristic impedance of the line.

For example, in the case of the T section line, the impedance $Z_0 \tanh \theta/2$, which is equal to A, may be reciprocated with respect to Z_0, giving $Z_0 \coth \theta/2 = A + 2B$. Thus any ladder network built up of numerical multiples of $Z_0 \tanh \theta/2$ and $Z_0 \coth \theta/2$ has a reciprocal with respect to Z_0 which can be physically realised.

In general, if any network of ladder form, associated with an artificial line section of constants Z_0 and θ, is constructed of numerical multiples of $Z_0 \tanh r\theta$ and $Z_0 \coth s\theta$, where r and s can have any integral values (and, in cases where bisection is possible, half integral numbers), then the network can be physically reciprocated with respect to Z_0.

Again, given a uniform transmission line having constants R, L, S, C, any ladder network made up of numerical multiples of $R + j\omega L$ and $\frac{1}{S + j\omega C}$ can be reciprocated with respect to the characteristic impedance, $\sqrt{\left(\frac{R + j\omega L}{S + j\omega C}\right)}$, of the uniform line, the reciprocals being $\frac{1}{S + j\omega C}$ and $R + j\omega L$ respectively.

§ 27. Equivalent T and *Π* Sections obtained by Reciprocation.

As a further example of the value of this method, suppose it is required to find a Π section artificial line having the same Z_0 as a

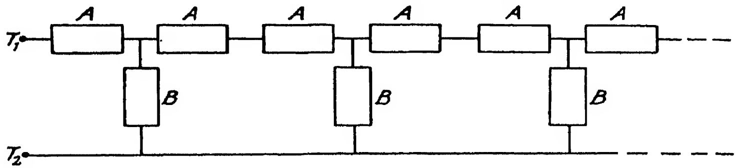


FIG. 59.

T section line. Let A and B be the elements of the T section, and A′, B′ those of the Π section.

Then Z_0 is the impedance of the infinite ladder network of Fig. 59.

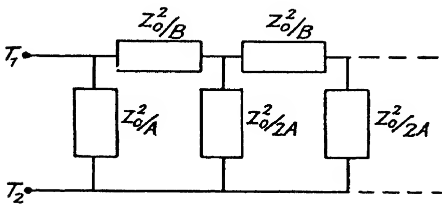


FIG. 60.

Suppose this infinite network is reciprocated with respect to Z_0; then the resulting network, which is shown in Fig. 60, has an impedance Z_0^2/Z_0, i.e. Z_0.

But this is an infinite Π section artificial line of which a single section is shown in Fig. 61(a). Thus the infinite Π section line of Fig. 60 constructed of individual Π sections, as in Fig. 61(a), has the same

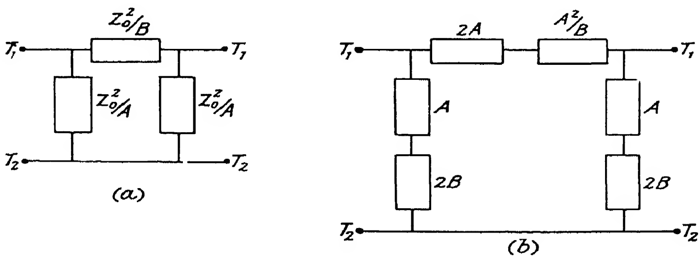


FIG. 61.

characteristic impedance, viz. Z_0, as the infinite T section line of Fig. 59; this can be readily checked by fundamental formula.

Replacing Z_0^2 by $A^2 + 2AB$,

$$Z_0^2/A = A + 2B$$

and $$Z_0^2/B = A^2/B + 2A.$$

Thus the Π section can be redrawn as in Fig. 61(*b*).

The question as to whether this Π section can actually be constructed depends on whether the reciprocal of B with respect to A is physically realisable.

For a T section in which A is a resistance and B is of ladder form there always exists a Π section having the same Z_0. It may also be proved that the Π section has the same propagation constant as the T section and is therefore exactly equivalent.

For example the Π section of Fig. 62(*b*) is equivalent to the T section of Fig. 62(*a*).

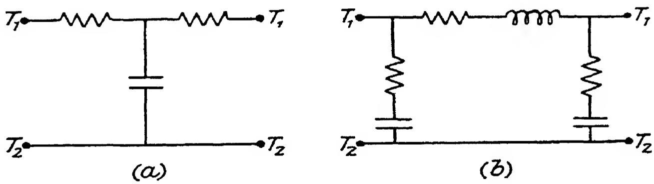


FIG. 62.

If R and C are the elements of Fig. 62(*a*) then the series element of Fig. 62(*b*) is a resistance 2R in series with an inductance CR^2 while each shunt element is a resistance R in series with a capacity C/2.

§ 28. A General Class of Artificial Line.

In Chapter II., § 18 (Fig. 43), a simple three-element artificial line was examined. This may be considered as a simple case of a more general type, of which a half section is shown in Fig. 63. A complete section is made by taking another identical half section and connecting T_1', T_2', etc., to T_1', T_2', etc., of the first half.

The half section consists of series impedances x_1, x_3, x_5, . . ., x_{2n-1}, of shunt impedances x_2, x_4, x_6, . . . , x_{2n}, and of cross impedances x_0', x_2', x_4', . . ., x'_{2n-2}.

Fig. 63 illustrates the case for $n = 3$.

Applying the Bisection Theorem, $Z_0 \coth \theta/2$ is the impedance of the network in Fig. 63 with the terminals T_1', T_2', T_3', etc., all

free; the network can in this case be redrawn as in Fig. 64(*a*). Similarly $Z_0 \tanh \theta/2$ is the impedance of the network of Fig. 63 with T_1', T_2', T_3', etc., all connected together; the network can be redrawn as in Fig. 64(*b*).

Let the impedances of x_1, x_3, x_5, etc., be a_1, a_3, a_5, etc., and of x_0', x_2, x_2', x_4, x_4', etc., be

$$\frac{1}{a_0'},\ \frac{1}{a_2},\ \frac{1}{a_2'},\ \frac{1}{a_4},\ \frac{1}{a_4'},\ \text{etc.}$$

Then

$$Z_0 \coth\frac{\theta}{2} = \frac{K(a_1, a_2, \ldots, a_{2n})}{K(a_2, a_3, \ldots, a_{2n})} \quad . \quad . \quad . \quad . \quad . \quad 28.1$$

$$Z_0 \tanh\frac{\theta}{2} = \frac{K\{a_1, (a_2 + a_2'), a_3, (a_4 + a_4'), \ldots, a_{2n-1}\}}{K\{a_0', a_1, (a_2 + a_2'), a_3, \ldots, a_{2n-1}\}}. \quad 28.2$$

These two equations determine the constants Z_0 and θ.

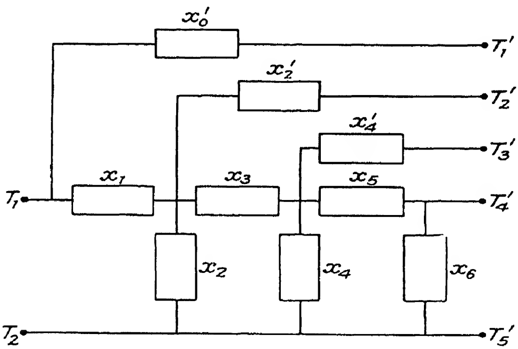


FIG. 63.

A special case is of interest and will be used later in Chapter VII.

Let x_1, x_3, x_5, etc., be each numerical multiples of one type of impedance X, so that a_1, a_3, a_5, etc., are respectively b_1X, b_3X, b_5X, etc.

Let all the other x's be numerical multiples of another impedance Y which is equal to Q^2/X, where Q is any impedance, so that a_0', a_2, a_2', etc., will be respectively

$$\frac{b_0'X}{Q^2},\quad \frac{b_2X}{Q^2},\quad \frac{b_2'X}{Q^2},\ \text{etc.}$$

Now choose the b's so that

$$\begin{aligned} b_1 &= b_0' \\ b_2 &= b_1 \\ b_3 &= b_2 + b_2' \\ b_4 &= b_3 \\ b_5 &= b_4 + b_4' \\ b_6 &= b_5 \qquad \text{etc.} \end{aligned}$$

If these values are inserted in 28.1 and 28.2 it will be seen on applying 22.9 that $Z_0{}^2$, the product of $Z_0 \coth \theta/2$ and $Z_0 \tanh \theta/2$, is equal to Q^2.

Suppose X is an inductance L, Q a resistance R, then Y is a capacity L/R^2. Thus a large class of artificial line sections can be

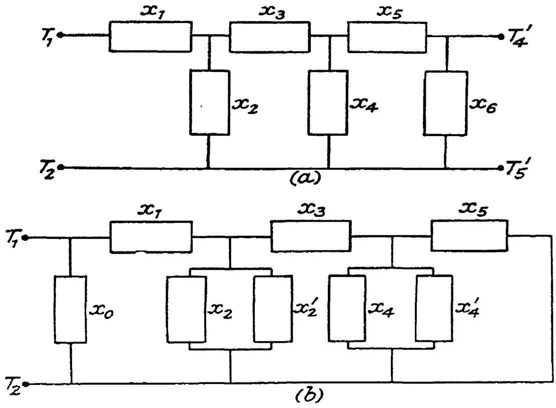


FIG. 64.

constructed of inductances and capacities, the characteristic impedance of any member of the class being a resistance at all frequencies. Such networks are of great importance and will be discussed in Chapter VII.

EXAMPLES AND NOTES.

1. Properties of the ladder network.

(a) The network $x_1, x_2, \ldots, x_n$ is electrically symmetrical, i.e. $\phi = 0$ if

$$K(a_1, a_2, \ldots, a_{n-1}) = K(a_2, a_3, \ldots, a_n).$$

(b) The voltage across x_n for a given input voltage will be independent of the value of x_n if

$$K(a_1, a_2, \ldots, a_{n-1}) = 0.$$

(*c*) The current through x_n for a given input voltage will be independent of the value of x_n if

$$K(a_1, a_2, \ldots, a_{n-2}) = 0,$$

and if this is so the receiving-end impedance will be

$$K(a_1, a_2, \ldots, a_{n-1}).$$

(*d*) The continuant $K(a_1, a_2, \ldots, a_n)$ can be expressed as

$$a_r K(a_1, a_2, \ldots, a_{r-1}) \,.\, K(a_{r+1}, a_{r+2}, \ldots, a_n) + K(a_1, a_2, \ldots, a_{r-2}, a_{r-1} + a_{r+1}, a_{r+2}, \ldots, a_n)$$

and is then of the form

$$a_r X + Y,$$

where X and Y are independent of a_r.

Hence the condition that the receiving-end impedance is independent of the value of an intermediate element x_r is either

$$K(a_1, a_2, \ldots, a_{r-1}) = 0,$$

or

$$K(a_{r+1}, a_{r+2}, \ldots, a_n) = 0.$$

(*e*) The current through x_n will be zero for all values of x_n if

$$K(a_1, a_2, \ldots, a_{n-1}) = \infty.$$

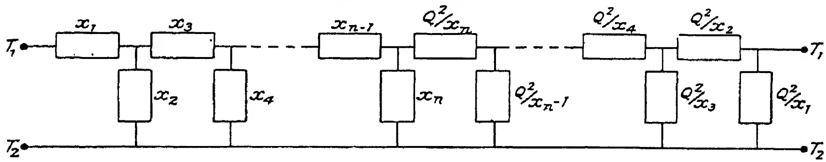


FIG. 65.

(*f*) Examine the significance of the zeros occurring in equations 22.6, 22.7 and 22.8.

2. Prove that a simple continuant is unaltered in value if any five consecutive terms,

$$a_r, a_{r+1}, a_{r+2}, a_{r+3}, a_{r+4},$$

are replaced by the three terms

$$a_r + \frac{a_{r+2}a_{r+3}}{K(a_{r+1}, a_{r+2}, a_{r+3})}, \; K(a_{r+1}, a_{r+2}, a_{r+3}), \; a_{r+4} + \frac{a_{r+2}a_{r+1}}{K(a_{r+1}, a_{r+2}, a_{r+3})},$$

and that this may be regarded as a statement of either the Star-Delta or Delta-Star transformation, according as to whether a_r stands for a shunt or series impedance.

3. Show (using continuants) that, for an asymmetric network constructed as in Fig. 65,

$$Z = Q.$$

4. Show that $Z_0 \sinh \theta$, the receiving-end impedance of the artificial line section of § 24, Fig. 52, short-circuited at the output, is

$$K(a_1, \ldots, a_n, \ldots, a_1).$$

Show also that

$$\frac{1}{K(a_2, \ldots, a_n, \ldots, a_2)} = \frac{Z_0}{\sinh\theta}$$

and that

$$K(a_1, \ldots, a_n, \ldots, a_2) = \cosh\theta.$$

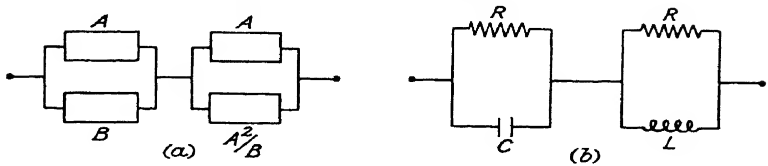


FIG. 66.

5. The impedance of the network of Fig. 66(a) is equal to A. A special case is that the impedance of the network of Fig. 66(b) is equal to R, if

$$L/C = R^2.$$

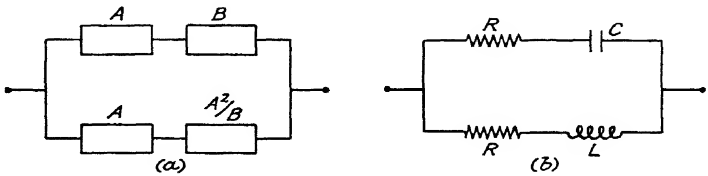


FIG. 67.

6. The impedance of the network of Fig. 67(a) is equal to A. A well-known special case is that the impedance of the network of Fig. 67(b) is equal to R, if

$$L/C = R^2.$$

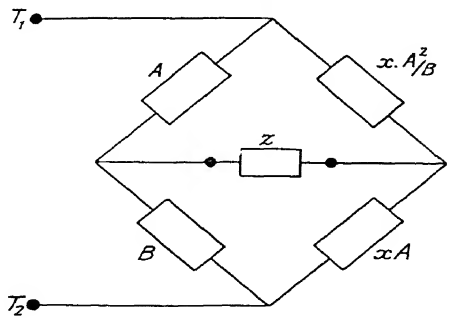


FIG. 68.

7. In the bridge network shown in Fig. 68, where x is any numerical quantity, z is conjugate to the terminals T_1T_2, i.e. if a voltage is applied to the terminals T_1T_2 no current will flow through z.

If A and B are resistances this is an ordinary Wheatstone bridge. If A is a resistance and B an inductance in series with a resistance, then A^2/B is a leaky capacity and the Maxwell inductance bridge is obtained.

Similarly, by making A a resistance and B any ladder network a whole class of aperiodic alternating current bridges is obtained.

8. Show that the T section of Fig. 69 is equivalent to a standard Π section,

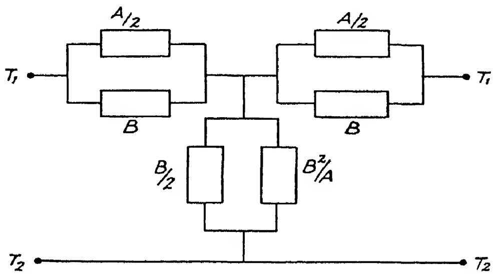


FIG. 69.

and, therefore, if in a Π section the shunt element is a resistance and the series member is of ladder form, there always exists a physically equivalent T.

9. Show that the T section of Fig. 70 is equivalent to two standard T sections, and that therefore if in a given T section B is a resistance and A is of ladder form it is always possible to construct a single T section equivalent to two of the original sections.

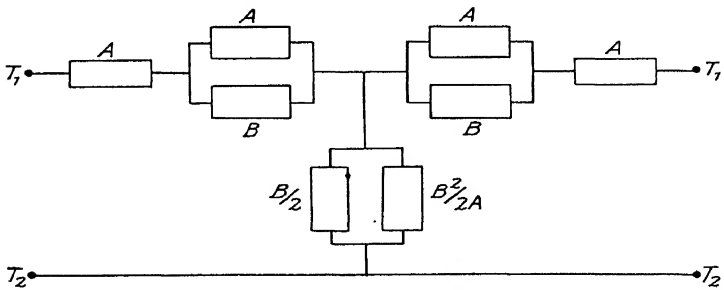


FIG. 70.

10. Show that the T section shown in Fig. 71 has a characteristic impedance which is rational and equal to

$$2B - \frac{AB}{A + B},$$

and show that the results of Examples 5 and 6 can be deduced as a simple property of this T section artificial line.

Obtain the corresponding Π section.

11. Given a T section having series and shunt elements A and B, in which A is a resistance, show that it is possible to construct an associated artificial

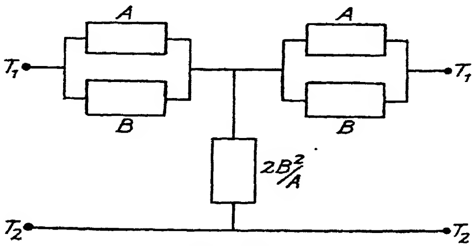


FIG. 71.

line whose characteristic impedance is $Z_0 \sinh \theta/2$, where Z_0 and θ are the constants of the original section.

12. If Z_1 is the sending-end impedance of an artificial line terminated by z, and Z_2 is the sending-end impedance of the same artificial line terminated by Z_0^2/z, then $Z_1Z_2 = Z_0^2$.

The equation 8.5 is a special case of this result.

CHAPTER IV.

EQUIVALENT NETWORKS RELATED TO ARTIFICIAL LINES.

§ 29. Simple Cases of Equivalent Networks.

In this chapter we shall deal with a class of equivalent networks related to artificial lines. Their practical value is probably not very great, and the whole chapter may be omitted without much loss. They are, however, of interest in that they show the remarkably intimate relations between artificial lines and hyperbolic functions,

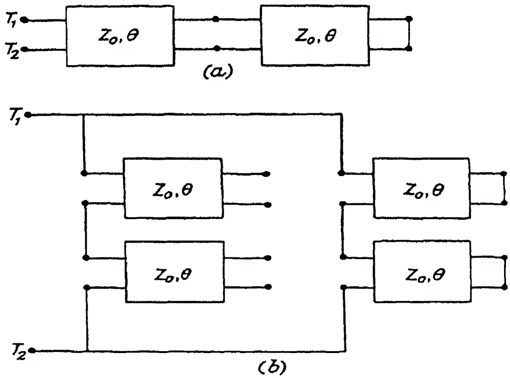


FIG. 72.

how almost any linear trigonometric formula relating the odd hyperbolic functions, i.e. sinh, tanh, and coth, has some physical interpretation in artificial line theory.

Let us start first with a simple formula

$$\tanh 2\theta = \frac{2 \tanh \theta}{1 + \tanh^2 \theta} \quad . \quad . \quad . \quad . \qquad 29.1$$

We may rewrite the right-hand side as

$$\frac{1}{\dfrac{1}{2\tanh\theta} + \dfrac{1}{2\coth\theta}}.$$

Multiplying each side by Z_0, we can write

$$Z_0 \tanh 2\theta = \frac{1}{\dfrac{1}{2Z_0\tanh\theta} + \dfrac{1}{2Z_0\coth\theta}}. \quad . \quad 29.2$$

Now if Z_0 and θ are the constants of an artificial line, consider the physical meaning of this equation.

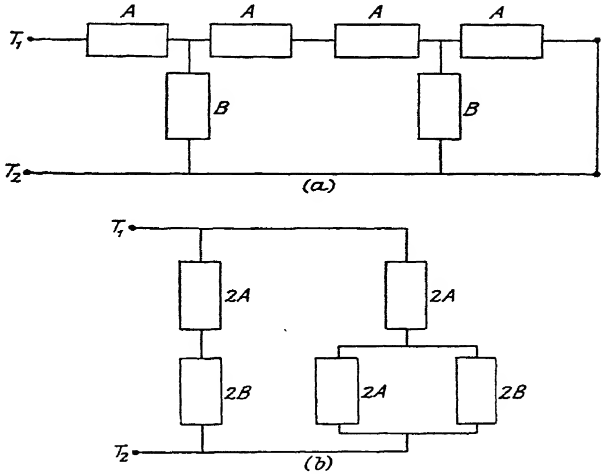


FIG. 73.

$Z_0 \tanh 2\theta$ is the impedance of a two-section artificial line short-circuited at the output. The right side of the equation is the impedance of the two impedances $2Z_0 \tanh\theta$ and $2Z_0 \coth\theta$ connected in parallel, which are respectively twice the impedance of a single section short- and open-circuited at the output.

Thus 29.2 states physically that the two networks shown in Fig. 72 have identical impedances, the rectangle representing any artificial line section whatever.

For a simple T section we obtain the two networks of Fig. 73, which have identical impedances.

The formula

$$\coth 2\theta = \frac{1 + \tanh^2 \theta}{2 \tanh \theta}$$
$$= \tfrac{1}{2} \coth \theta + \tfrac{1}{2} \tanh \theta \quad . \quad . \quad 29.3$$

gives

$$Z_0 \coth 2\theta = \tfrac{1}{2} Z_0 \coth \theta + \tfrac{1}{2} Z_0 \tanh \theta \quad . \quad . \quad 29.4$$

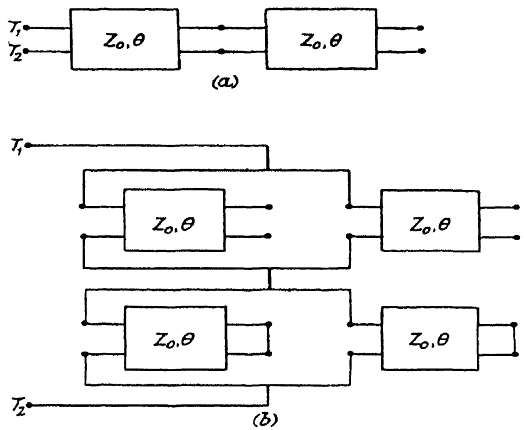


FIG. 74.

and thus, corresponding to Fig. 72, we have the impedance identity of Fig. 74.

We may note in passing that the network of Fig. 74(*b*) can be obtained from that of Fig. 72(*b*) by reciprocating with respect to Z_0.

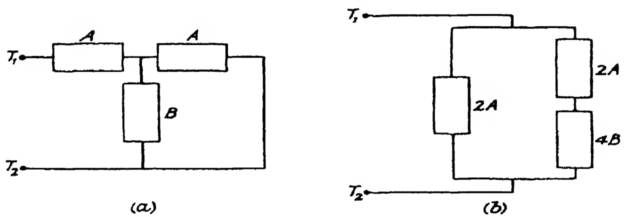


FIG. 75.

These two cases are not even the simplest, for we may express $Z_0 \tanh \theta$ in terms of $Z_0 \tanh \theta/2$ and $Z_0 \coth \theta/2$. Thus for a single T section the application of 29.2 gives the equivalence of the networks shown in Fig. 75.

The truth of this can be at once proved by writing down the impedances both of which are A + AB/(A + B).

The example shown in Fig. 73 is for simple T sections, but the identity holds for any artificial line section however complex.

§ 30. General Formulæ.

The simple cases of equivalent networks given in the previous section suggest that they are simple cases of more general results. An examination of formulæ for expansion of hyperbolic functions shows that many exist which lead to equivalent networks.

Among the most important are the following:

$$\operatorname{Tanh} n\theta = \frac{n \tanh \theta}{1 +}\,\frac{(n^2 - 1^2)\tanh^2 \theta}{3 +}\,\frac{(n^2 - 2^2)\tanh^2\theta}{5 +}\ldots, \qquad 30.1$$

$$\operatorname{Tanh} n\theta = \sum_{r=0}^{r=\frac{n}{2}-1} \frac{\tanh\theta \, . \, \dfrac{2}{n \sin^2 \dfrac{2r+1}{2n} . \pi}}{1 + \dfrac{\tanh^2\theta}{\tan^2 \dfrac{2r+1}{2n} . \pi}}$$

when n is even . . 30.2

$$\operatorname{Tanh} n\theta = \frac{\tanh\theta}{n} + \sum_{r=0}^{r=\frac{n-3}{2}} \frac{\tanh\theta \, . \, \dfrac{2}{n \sin^2 \dfrac{2r+1}{2n} . \pi}}{1 + \dfrac{\tanh^2\theta}{\tan^2 \dfrac{2r+1}{2n} . \pi}}$$

when n is odd . . 30.3

$$\operatorname{Coth} n\theta = \frac{1}{n}(\tanh\theta + \coth\theta) + \sum_{r=1}^{r=\frac{n}{2}-1} \frac{\tanh\theta \, . \, \dfrac{2}{n \sin^2 \dfrac{r\pi}{n}}}{1 + \dfrac{\tanh^2\theta}{\tan^2 \dfrac{r\pi}{n}}}$$

when n is even . . 30.4

$$\operatorname{Coth} n\theta = \frac{\coth\theta}{n} + \sum_{r=1}^{r=\frac{n-1}{2}} \frac{\tanh\theta \, . \, \dfrac{2}{n \sin^2 \dfrac{r\pi}{n}}}{1 + \dfrac{\tanh^2\theta}{\tan^2 \dfrac{r\pi}{n}}}$$

when n is odd, . 30.5

$$\operatorname{Tanh}\frac{n\theta}{2} = \frac{1}{n}\sum_{r=0}^{r=\frac{n}{2}-1}\frac{1}{\dfrac{\sin^2\dfrac{2r+1}{2n}\,.\,\pi}{\sinh\theta} + \frac{1}{2}\tanh\dfrac{\theta}{2}}$$

when n is even, . 30.6

$$\operatorname{Coth}\frac{n\theta}{2} = \frac{1}{n}\left\{2\coth\theta + \sum_{r=1}^{r=\frac{n}{2}-1}\frac{1}{\dfrac{\sin^2\dfrac{r\pi}{n}}{\sinh\theta} + \frac{1}{2}\tanh\dfrac{\theta}{2}}\right\}$$

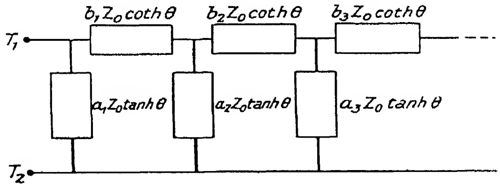
when n is even, . 30.7

$$\operatorname{Tanh}\frac{n\theta}{2} = \frac{1}{n}\left\{\tanh\frac{\theta}{2} + \sum_{r=0}^{r=\frac{n-3}{2}}\frac{1}{\dfrac{\sin^2\dfrac{2r+1}{2n}\,.\,\pi}{\sinh\theta} + \frac{1}{2}\tanh\dfrac{\theta}{2}}\right\}$$

when n is odd, . 30.8

$$\operatorname{Coth}\frac{n\theta}{2} = \frac{1}{n}\left\{\coth\frac{\theta}{2} + \sum_{r=1}^{r=\frac{n-1}{2}}\frac{1}{\dfrac{\sin^2\dfrac{r\pi}{n}}{\sinh\theta} + \frac{1}{2}\tanh\dfrac{\theta}{2}}\right\}$$

when n is odd, . 30.9



$$a_1 = n. \qquad b_1 = \frac{3n}{n^2 - 1}.$$

$$a_2 = \frac{n(n^2 - 2^2)}{5(n^2 - 1^2)}. \qquad b_2 = \frac{7n(n^2 - 2^2)}{(n^2 - 1^2)(n^2 - 3^2)}.$$

$$a_3 = \frac{n(n^2 - 2^2)(n^2 - 4^2)}{9(n^2 - 1^2)(n^2 - 3^2)}. \qquad b_3 = \frac{11n(n^2 - 2^2)(n^2 - 4^2)}{(n^2 - 1^2)(n^2 - 3^2)(n^2 - 5^2)}.$$

etc. etc.

Fig. 76.—Equivalent circuit for $Z_0 \tanh n\theta$.

First consider 30.1; from the finite continued fraction

$$\tanh n\theta = \frac{n \tanh \theta}{1 +} \frac{(n^2 - 1^2) \tanh^2 \theta}{3 +} \frac{(n^2 - 2^2) \tanh^2 \theta}{5 +} \cdots$$

we get

$$Z_0 \tanh n\theta = \frac{1}{1/nZ_0 \tanh \theta +} \frac{1}{\dfrac{3n}{n^2 - 1^2} \cdot Z_0 \coth \theta +} \frac{1}{1 \Big/ \dfrac{n(n^2 - 2^2)}{5(n^2 - 1^2)} \cdot Z_0 \tanh \theta +} \cdots \qquad 30.10$$

If we compare this continued fraction with the continued fraction for the impedance of a ladder network (Chap. III., § 23)

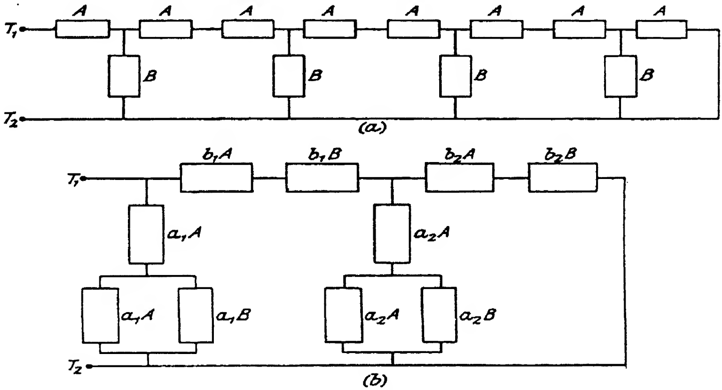


FIG. 77.

we see at once that 30.10 is a statement that the impedance of the ladder network shown in Fig. 76 is $Z_0 \tanh n\theta$.

Thus in the case of four T sections short-circuited at the output, as shown in Fig. 77 (a), the values of the a's and b's are

$$a_1 = 4,$$

$$a_2 = \frac{4(4^2 - 2^2)}{5(4^2 - 1)} = \frac{16}{25},$$

$$b_1 = \frac{3 \times 4}{4^2 - 1} = \frac{4}{5},$$

$$b_2 = \frac{7 \times 4 \times (4^2 - 2^2)}{(4^2 - 1^2)(4^2 - 3^2)} = \frac{16}{5},$$

and therefore since

$$Z_0 \coth \theta = A + B$$

and

$$Z_0 \tanh \theta = A + \frac{AB}{A + B},$$

the network of Fig. 77(*b*) is equivalent to that of Fig. 77(*a*).

By reciprocating the network of Fig. 76 with respect to Z_0 we obtain the equivalent circuit for $Z_0 \coth n\theta$, shown in Fig. 78, where the a's and b's have the same values as in Fig. 76:

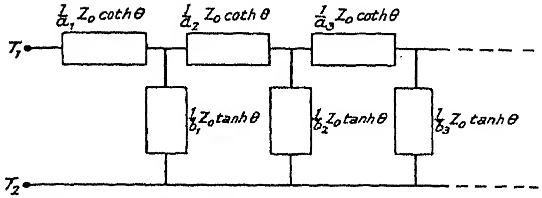


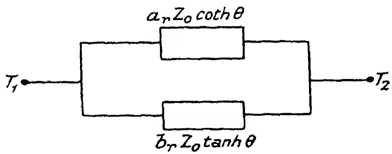
FIG. 78.—Equivalent circuit for $Z_0 \coth n\theta$.

Consider now the identity 30.2; we have, when n is even,

$$Z_0 \tanh n\theta = \sum_{r=0}^{r=\frac{n}{2}-1} \frac{Z_0 \tanh \theta \cdot \dfrac{2}{n \sin^2 \dfrac{2r+1}{2n} \cdot \pi}}{1 + \dfrac{\tanh^2 \theta}{\tan^2 \dfrac{2r+1}{2n} \cdot \pi}}$$

$$= \sum_{r=0}^{r=\frac{n}{2}-1} \frac{1}{\dfrac{1}{Z_0 \tanh \theta \cdot \dfrac{2}{n \sin^2 \dfrac{2r+1}{2n} \cdot \pi}} + \dfrac{1}{Z_0 \coth \theta \cdot \dfrac{2}{n \cos^2 \dfrac{2r+1}{2n} \cdot \pi}}} \qquad 30.11$$

But the expression under the Σ is the impedance of the simple circuit shown in Fig. 79.



$$a_r = \frac{2}{n \cos^2 \dfrac{2r+1}{2n} \cdot \pi}$$

$$b_r = \frac{2}{n \sin^2 \dfrac{2r+1}{2n} \cdot \pi}.$$

FIG. 79.

Hence the network in Fig. 80, consisting of $\frac{n}{2}$ pairs of multiples of $Z_0 \coth \theta$ and $Z_0 \tanh \theta$ has an impedance $Z_0 \tanh n\theta$.

From the identity 30.3 we can obtain an equivalent network of the same general type when n is odd.

Identities 30.4 and 30.5 yield equivalent networks for $Z_0 \coth n\theta$.

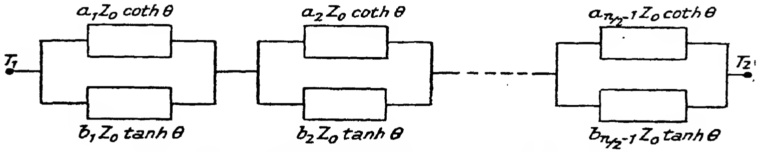


FIG. 80.—Equivalent circuit for $Z_0 \tanh n\theta$.

From these networks we can obtain by reciprocation a complete set of networks of a different type for both $Z_0 \tanh n\theta$ and $Z_0 \coth n\theta$.

For example, the impedance of the network of Fig. 80 is $Z_0 \tanh n\theta$—reciprocating with respect to Z_0 we have the network shown in Fig. 81 whose impedance is $Z_0 \coth n\theta$, the a's and b's having the same values as in Figs. 79 and 80.

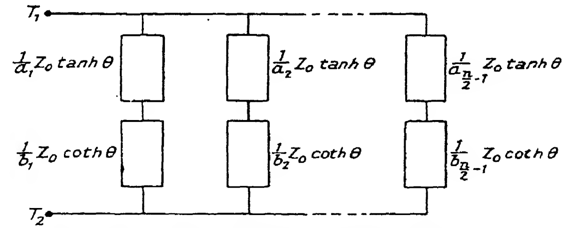


FIG. 81.—Equivalent circuit for $Z_0 \coth n\theta$.

For $Z_0 \coth n\theta$ we can get, from identities 30.4 and 30.5, equivalent networks of the general form of Fig. 80, hence, by reciprocating with respect to Z_0, networks of the general form of Fig. 81 can be obtained for $Z_0 \tanh n\theta$.

Identities 30.6, 30.7, 30.8, and 30.9 can be applied to T and Π sections, since they require the physical reality of $Z_0 \sinh \theta$ and $Z_0 / \sinh \theta$, but they cannot be applied to artificial lines in general.

The identities can be used to give many other different equivalent networks. For example, suppose $n = n_1 \times n_2$, then we can expand $\tanh n\theta$ either as $\tanh n_1(n_2\theta)$ or as $\tanh n_2(n_1\theta)$, obtaining networks of the forms shown in Figs. 78, 80 and 81, the impedances being numerical multiples of $Z_0 \tanh n_2\theta$ and $Z_0 \coth n_2\theta$ in the first case and of $Z_0 \tanh n_1\theta$ and $Z_0 \coth n_1\theta$ in the second.

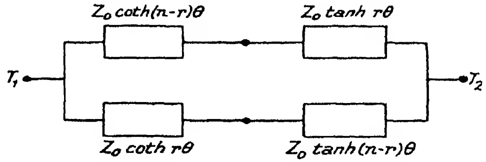


FIG. 82.—Equivalent circuit for $Z_0 \tanh n\theta$.

Again,

$$Z_0 \tanh n\theta = Z_0 \tanh \{(n-r)\theta + r\theta\} \qquad (0 < r < n)$$

$$= \frac{1}{\dfrac{1}{Z_0 \coth (n-r)\theta + Z_0 \tanh r\theta} + \dfrac{1}{Z_0 \tanh (n-r)\theta + Z_0 \coth r\theta}},$$

which is the impedance of the network of Fig. 82.

It can also be shown that the network of Fig. 83 is equivalent to $Z_0 \tanh n\theta$.

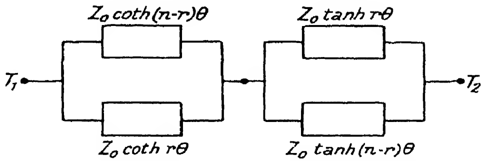


FIG. 83.—Equivalent circuit for $Z_0 \tanh n\theta$.

It will be seen that Fig. 83 differs from Fig. 82 in that the junction points of the two pairs of impedances in each arm have been directly connected together; in fact in Fig. 82 these two points are conjugate with respect to the pair of terminals T_1T_2.

Most of the equivalent networks given in this chapter are special cases of more general properties of ladder networks consisting of two different kinds of impedances: these properties are discussed in a long and interesting paper by W. Cauer, *Archiv für Elektrotechnik,* Heft 4, Band XVII., 1926.

CHAPTER V.

ARTIFICIAL LINES RELATED TO THE UNIFORM TELEPHONE OR TRANSMISSION LINE.

§ 31. Relation of Artificial Lines to Uniform Lines.

In this chapter we shall consider a number of important and interesting artificial lines related to the uniform line. They come under four heads:—

1. *Artificial Telephone Lines.*—Networks having a pair of input terminals and a pair of output terminals that can replace to a sufficient degree of approximation a length of uniform line as far as measurements at the two ends are concerned.

2. *Line Balances for Uniform Lines.*—Networks with a single pair of terminals having an impedance equal at all frequencies to the characteristic impedance of the uniform line.

3. *Impedance Corrective Networks.*—Networks with a single pair of terminals which, when associated with the infinite uniform line, reduce its input impedance to a non-inductive resistance.

4. *Distortion Corrective Networks.*—Networks having two pairs of terminals which when used in conjunction with the uniform line either reduce or eliminate the distortion due to variation with frequency of the attenuation and phase constants of the uniform line.

§ 32. Artificial Telephone Lines.

An artificial line designed to be equivalent to a telephone line both as regards input and output terminals is called an *Artificial Telephone Line.*

The usual way of constructing these is that used in "Standard Cable" boxes; if R, L, S and C are the constants of the line per mile, T sections are made up as in Fig. 84, and this is taken as equivalent to a length l of the original line.

The validity and degree of approximation of this arrangement must be examined.

Let Z_0' and θ be the constants of this T section; Z_0 and P, the

characteristic impedance and propagation constant of the uniform line, are given by

$$Z_0 = \sqrt{\left(\frac{R + j\omega L}{S + j\omega C}\right)} \text{ and } P = \sqrt{\{(R + j\omega L)(S + j\omega C)\}}.$$

Then in the T section of Fig. 84,

$$A = \frac{l}{2}(R + j\omega L)$$

$$B = \frac{1}{l(S + j\omega C)}.$$

$$\text{Hence } Z_0' = \sqrt{\left\{\frac{l^2}{4}(R + j\omega L)^2 + \frac{R + j\omega L}{S + j\omega C}\right\}}$$

$$= \sqrt{\left(\frac{R + j\omega L}{S + j\omega C}\right)}\sqrt{\left\{1 + \frac{l^2}{4}(R + j\omega L)(S + j\omega C)\right\}}$$

$$= Z_0\sqrt{\left(1 + \left[\frac{Pl}{2}\right]^2\right)} \quad . \quad . \quad . \quad . \quad . \quad 32.1$$

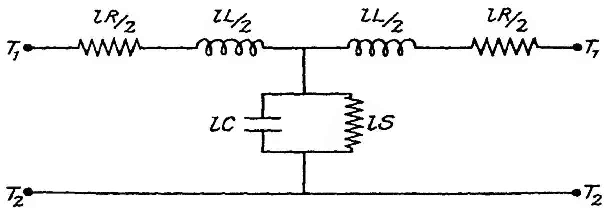


FIG. 84.

Expanding by the Binomial Theorem

$$Z_0' = Z_0\left(1 + \frac{P^2l^2}{8}\right) \text{ approx.,} \quad . \quad . \quad 32.2$$

provided that Pl is small.

The propagation constant θ can also be expressed in terms of Pl, for

$$Z_0' \tanh\frac{\theta}{2} = \frac{l}{2}(R + j\omega L)$$

$$\tanh\frac{\theta}{2} = \frac{l}{2}\cdot\frac{R + j\omega L}{Z_0'}$$

$$= \frac{l}{2}\cdot\frac{R + j\omega L}{Z_0\sqrt{\left[1 + \left(\frac{Pl}{2}\right)^2\right]}}$$

$$= \frac{Pl}{2\sqrt{\left(1 + \frac{P^2l^2}{4}\right)}}.$$

Hence, using the expansion

$$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots,$$

we obtain

$$\theta = 2 \tanh^{-1} \frac{Pl}{2\sqrt{\left(1 + \frac{P^2l^2}{4}\right)}} \quad . \quad . \quad . \quad . \quad 32.3$$

$$= \frac{Pl}{\sqrt{\left(1 + \frac{P^2l^2}{4}\right)}} + \frac{1}{3.4} \frac{P^3l^3}{\left(1 + \frac{P^2l^2}{4}\right)^{3/2}} + \ldots$$

Expanding and neglecting terms of higher order than third in Pl,

$$\theta = Pl + P^3l^3\left(\frac{1}{3.4} - \frac{1}{2.4}\right)$$

$$= Pl\left(1 - \frac{P^2l^2}{24}\right) \text{ approximately.} \quad . \quad . \quad 32.4$$

By keeping l small, Z_0' and θ can be made to approximate as nearly as may be desired to Z_0 and Pl; for this reason in "Standard Cable" boxes each individual section is made up to replace not more than two miles. A "Standard Cable" block for twenty miles, for example, would be built up of ten two-mile sections.

Equations 32.2 and 32.4 allow the degree of error in an artificial telephone line of this type to be easily calculated.

For example, in the case of a two-mile section "Standard Cable" at 800 cycles,

$$Pl = 0{\cdot}3\underline{/45^\circ} \text{ approximately,}$$

$$P^2l^2 = 0{\cdot}09\underline{/90^\circ},$$

$$\frac{Z_0'}{Z_0} = 1 + 0{\cdot}011\underline{/90^\circ},$$

and

$$\frac{\theta}{Pl} = 1 - 0{\cdot}0038\underline{/90^\circ}.$$

The errors therefore are so small as to be negligible.

§ 33. Artificial Telephone Lines. Another Method.

This method is based on the equivalent bridge described in Chapter II., § 17. To construct the equivalent bridge for a length l

of the uniform line it is necessary to construct physical networks to have impedances equal to

$$Z_0 \tanh \frac{Pl}{2}$$

$$\text{and} \quad Z_0 \coth \frac{Pl}{2}.$$

It happens that there are a number of ways of doing this to any desired degree of approximation.

Consider first $Z_0 \tanh \frac{Pl}{2}$; for simplicity let

$$R + j\omega L = X$$

$$\frac{1}{S + j\omega C} = Y$$

so that

$$Z_0 \tanh \frac{Pl}{2} = \sqrt{(XY)} \,.\, \tanh \frac{l}{2}\sqrt{\frac{X}{Y}}.$$

There is a well-known expansion of $\tanh x$ as an infinite continued fraction due to Euler, viz.

$$\tanh x = \frac{x}{1} + \frac{x^2}{3} + \frac{x^2}{5} + \ldots$$

$$= \cfrac{1}{1/x + \cfrac{1}{3/x + \cfrac{1}{5/x + \cfrac{1}{\ldots}}}}$$

Thus

$$\tanh \frac{Pl}{2} = \tanh \frac{l}{2}\sqrt{\frac{X}{Y}}$$

$$= \cfrac{1}{\frac{2}{l}\sqrt{\frac{Y}{X}} + \cfrac{1}{\frac{3.2}{l}\sqrt{\frac{Y}{X}} + \cfrac{1}{\frac{5.2}{l}\sqrt{\frac{Y}{X}} + \cfrac{1}{\ldots}}}}$$

and therefore, multiplying by $\sqrt{XY}$,

$$Z_0 \tanh \frac{Pl}{2} = \cfrac{1}{\frac{2}{lX} + \cfrac{1}{\frac{3.2Y}{l} + \cfrac{1}{\frac{5.2}{lX} + \cfrac{1}{\ldots}}}}$$

But this is readily seen to be the impedance of the network of Fig. 85, where

$$s_1 = \frac{lX}{2},$$

and generally
$$s_n = \frac{lX}{2(4n-3)},$$

$$t_1 = \frac{6Y}{l},$$

and generally
$$t_n = \frac{Y}{l}2(4n-1).$$

But
$$\frac{lX}{2(4n-3)} = \frac{l(R+j\omega L)}{2(4n-3)}$$

and is the impedance of a resistance of value

$$\frac{lR}{2(4n-3)}$$

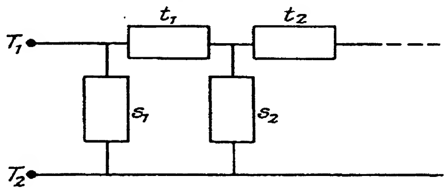


FIG. 85.

connected in series with an inductance

$$\frac{lL}{2(4n-3)}.$$

Similarly $\frac{Y}{l}2(4n-1)$ is the impedance of a capacity of value $\frac{lC}{2(4n-1)}$ shunted by a leak of value $\frac{lS}{2(4n-1)}$.

Thus $Z_0 \tanh Pl/2$ can be represented by the infinite network of Fig. 86, where

$$L_n = \frac{lL}{2(4n-3)},$$

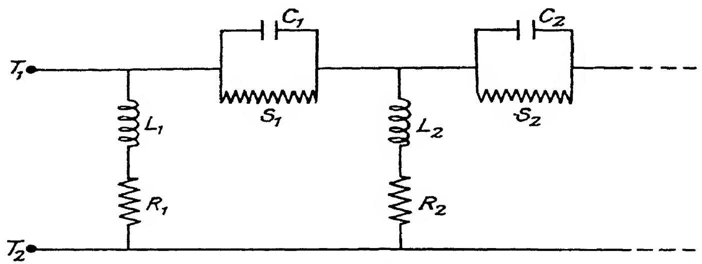
$$R_n = \frac{lR}{2(4n-3)},$$

$$C_n = \frac{lC}{2(4n-1)},$$

$$S_n = \frac{lS}{2(4n-1)}.$$

Another infinite network can be obtained by using the expansion

$$\tanh x = \sum_{n=1}^{n=\infty} \frac{8x}{(2n - 1)^2\pi^2 + 4x^2}.$$



$Z_0 \tanh Pl/2$.
FIG. 86.

Thus

$$Z_0 \tanh \frac{Pl}{2} = \sqrt{XY} \tanh \frac{l}{2}\sqrt{\frac{X}{Y}}$$

$$= \sum_{n=1}^{\infty} \frac{\sqrt{XY}\, 8\, \frac{l}{2}\sqrt{\frac{X}{Y}}}{(2n - 1)^2\pi^2 + l^2\frac{X}{Y}}$$

$$= \sum_{n=1}^{\infty} \frac{1}{\frac{(2n - 1)^2\pi^2}{4lX} + \frac{l}{4Y}}.$$

This is the impedance of the infinite network of Fig. 87,

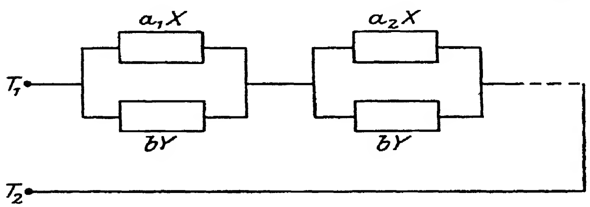


FIG. 87.

where $a_n = 4l/(2n - 1)^2\pi^2$, $b = 4/l$.

Replacing X and Y by $R + j\omega L$ and $\dfrac{1}{S + j\omega C}$ we have the result that $Z_0 \tanh Pl/2$ is the impedance of the infinite network of Fig. 88, where

$$R_n = \frac{4lR}{(2n-1)^2\pi^2},$$

$$L_n = \frac{4lL}{(2n-1)^2\pi^2},$$

$$C' = \frac{lC}{4},$$

$$S' = \frac{lS}{4}.$$

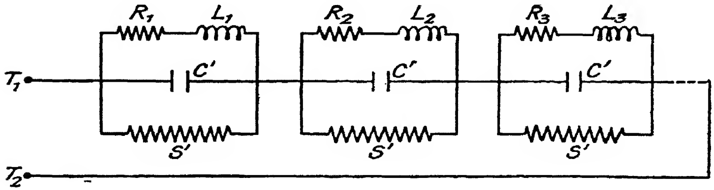


FIG. 88.

Yet another network equivalent to $Z_0 \tanh Pl/2$ can be obtained from the expansion

$$\coth x = \frac{1}{x} + \sum_{n=1}^{n=\infty} \frac{2x}{x^2 + n^2\pi^2},$$

$$Z_0 \tanh \frac{Pl}{2} = \frac{Z_0}{\coth \frac{Pl}{2}} = \frac{\sqrt{XY}}{\coth \frac{l}{2}\sqrt{\frac{X}{Y}}}$$

$$= \sqrt{XY} \cdot \frac{1}{\dfrac{1}{\frac{l}{2}\sqrt{\frac{X}{Y}}} + \displaystyle\sum_{n=1}^{\infty} \frac{l\sqrt{\frac{X}{Y}}}{\frac{l^2}{4}\cdot\frac{X}{Y} + n^2\pi^2}}$$

$$= \frac{1}{\dfrac{1}{lX/2} + \displaystyle\sum_{n=1}^{\infty} \frac{1}{\frac{lX}{4} + \frac{n^2\pi^2 Y}{l}}},$$

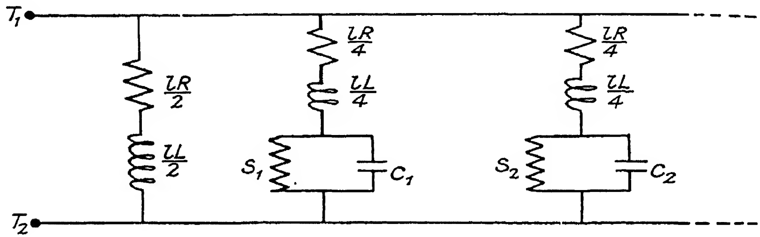
which is the impedance of the infinite network of Fig. 89, where

$$S_n = \frac{lS}{n^2\pi^2},$$

$$C_n = \frac{lC}{n^2\pi^2}.$$

We have found three separate solutions of the problem in the form of infinite networks, and by taking sufficient terms any required degree of approximation can be obtained.

A similar set of equivalent networks for $Z_0 \coth Pl/2$ can be worked out in the same way from the same trigonometric formulæ. The Method of Reciprocation allows them to be obtained in a very simple and direct way by reciprocating $Z_0 \tanh Pl/2$ with respect to Z_0; for it is seen that since $Z_0^2 = XY$, X reciprocates into Y and Y into X.



$Z_0 \tanh Pl/2$.

FIG. 89.

In practice, of course, the infinite networks cannot be obtained, but approximation can be made to any required degree by building the networks out far enough. If, moreover, the arms $Z_0 \tanh Pl/2$ and $Z_0 \coth Pl/2$ are made up of those networks reciprocal to one another with respect to Z_0, and are built out to the same degree of approximation, they will still be accurately reciprocal with respect to Z_0. The resulting artificial line will have its characteristic impedance accurately equal to Z_0 for the uniform line, all the error being thrown into the propagation constant.

The equivalent bridge artificial line has many points of theoretical interest. Though not strictly coming within the scope of this book it may be mentioned that the equivalent networks for

$$Z_0 \tanh Pl/2 \quad \text{and} \quad Z_0 \coth Pl/2$$

have very interesting connections with the theory of Normal Modes of Vibration and with their corresponding mechanical and acoustic analogies.

Indeed the networks of Fig. 89 and the reciprocal of Fig. 88 may be regarded as statements in physical form of the Heaviside Expansion for input current when a voltage is applied to the input terminals of a length $l/2$ of uniform line short- or open-circuited respectively at the far end.

It will also be seen that these networks may be regarded as limiting cases of the results of Chapter IV.

Just as it was pointed out in Chapter II. that the bridge section is more flexible than the T or Π sections, so the artificial telephone line just obtained is more flexible than the original uniform line. In a uniform line we are limited from physical reasons to a series impedance which must be an inductance in series with a resistance, and to a shunt impedance which must be a leaky capacity. But in the artificial telephone line just described there is no such limitation, for it will be seen, on referring back, that the physical existence of the equivalent networks for $Z_0 \tanh Pl/2$ and $Z_0 \coth Pl/2$ depends only on X and Y being physically realisable; it does not in any way depend on the fact that $X = R + j\omega L$ and $Y = \dfrac{1}{S + j\omega C}$.

Thus it is possible to construct an artificial line equivalent to any hypothetical uniform line, provided only that its shunt and series impedances are of a physically realisable form.

For example, it is only necessary to interchange X and Y throughout to obtain an artificial line having a characteristic impedance and propagation constant equal to

$$\sqrt{\frac{R + j\omega L}{S + j\omega C}} \quad \text{and} \quad \frac{1}{\sqrt{(R + j\omega L)(S + j\omega C)}}$$

respectively. This artificial line would be equivalent to a uniform line having per unit length a series impedance of $\dfrac{1}{S + j\omega C}$ and a shunt impedance $(R + j\omega L)$; but whereas the artificial line could be physically constructed to any desired degree of approximation, it appears to be impossible to construct or even to conceive of the construction of a uniform telephone line characterised by distributed series capacity and a distributed shunt inductance.

§ 34. Line Balances. T Sections.

The problem of obtaining a line balance for a uniform line would be solved if we could construct a T section artificial line having its characteristic impedance identically equal to that of the uniform line; for, if θ were the propagation constant of the artificial line, the sending-end impedance of n sections open- or short-circuited at the output would be $Z_0 \coth n\theta$ or $Z_0 \tanh n\theta$, and by making n large enough $\coth n\theta$ and $\tanh n\theta$ could be made to approximate to unity to any required degree.

It happens that simple T sections satisfying these requirements can be easily found.

Consider first the case when $S = 0$. In order that a T section shall have exactly the same characteristic impedance as the uniform line it is necessary to put $\sqrt{\dfrac{R + j\omega L}{j\omega C}}$ in the form $\sqrt{(A^2 + 2AB)}$, with the condition that A and B can be physically realised.

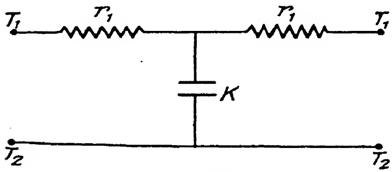


FIG. 90.

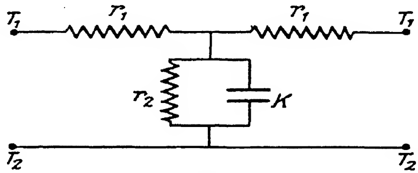


FIG. 91.

Now

$$\frac{R + j\omega L}{j\omega C} = \frac{L}{C}\left(1 + \frac{R}{j\omega L}\right)$$
$$= \left(\sqrt{\frac{L}{C}}\right)^2 + 2\sqrt{\frac{L}{C}} \cdot \frac{R}{2j\omega\sqrt{LC}},$$

which is of the form $A^2 + 2AB$, where

$$A = \sqrt{\frac{L}{C}},$$
$$B = \frac{R}{2j\omega\sqrt{LC}}.$$

But $\sqrt{\dfrac{L}{C}}$ has the dimensions of a resistance and $\dfrac{2\sqrt{LC}}{R}$ has the dimensions of a capacity.

Hence the simple T section of Fig. 90, where

$$r_1 = \sqrt{\frac{L}{C}}$$

and
$$K = \frac{2\sqrt{LC}}{R}$$

satisfies the requirements and can be used as a line balance for a leakless uniform line. The error involved in using n sections can be determined by calculating tanh $n\theta$ or coth $n\theta$.

When $S \neq 0$, there are two cases according as RC > or < LS as illustrated in Figs. 91 and 92:

Case I. RC > LS (Fig. 91).

$$r_1 = \sqrt{\frac{L}{C}},$$

$$r_2 = \frac{1}{2}\left(\sqrt{\frac{L}{C}}\right)\left(\frac{RC}{LS} - 1\right),$$

$$K = 2C\frac{\sqrt{LC}}{RC - LS}.$$

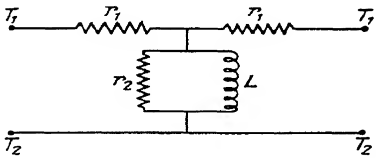


FIG. 92.

Case II. RC < LS (Fig. 92).

$$r_1 = \sqrt{\frac{R}{S}},$$

$$r_2 = \frac{1}{2}\left(\sqrt{\frac{R}{S}}\right)\left(\frac{LS}{RC} - 1\right),$$

$$L_1 = \frac{1}{2}\left(\sqrt{\frac{1}{RS}}\right)\left(\frac{LS - RC}{S}\right).$$

As an example take the case of a 200 lb. copper air line having the following constants per mile:

$$R = 8{\cdot}8 \text{ ohms.}$$
$$S = 10^{-6} \text{ mho.}$$
$$C = 8{\cdot}6 \times 10^{-3}\ \mu\text{F.}$$
$$L = 3{\cdot}66 \times 10^{-3} \text{ henry.}$$

A line balance can be constructed of T sections such as Fig. 91, where

$$r_1 = 652 \text{ ohms.}$$
$$r_2 = 6420 \text{ ohms.}$$
$$K = 1{\cdot}34\ \mu F.$$

Tanh $n\theta$ for this artificial line has been calculated for $n = 1, 2, 3$, and 6, and for $\omega = 2000$ and $\omega = 12{,}000$, corresponding approximately to frequencies of 300 and 2000 cycles per second. The results are shown in the following table and indicate how soon a high degree of approximation can be obtained:

	$\omega = 2000$ (300 ∿ approx.).	$\omega = 12{,}000$. (2000 ∿ approx.).
tanh θ	$1{\cdot}06\ \underline{/5^\circ\ 24'}$	$1{\cdot}0045\ \overline{\diagdown 3'}$
tanh 2θ	$1{\cdot}0034\ \overline{\diagdown 19'}$	$0{\cdot}99913\ \overline{\diagdown 0{\cdot}86''}$
tanh 3θ	$0{\cdot}9996\ \overline{\diagdown 5''}$	Very nearly unity
tanh 6θ	$0{\cdot}999999\ \overline{\diagdown 0{\cdot}004''}$	,, ,,

§ 35. Line Balances. Π Sections.

In a manner very similar to that used for the T section line balances, a series of Π sections can be obtained. Here it is necessary to express $\sqrt{\left(\dfrac{R + j\omega L}{S + j\omega C}\right)}$ in the form $\dfrac{AB}{\sqrt{(A^2 + 2AB)}}$, with the condition that A and B can be physically realised.

There are three cases which are shown in Figs. 93, 94 and 95.

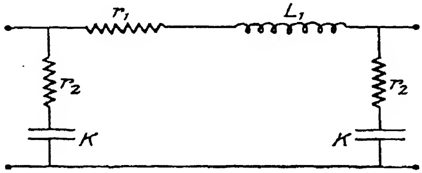


Fig. 93.

Case I. S = 0 (Fig. 93).

$$r_1 = 2\sqrt{\frac{L}{C}},$$

$$L_1 = 2\frac{L}{R} \cdot \sqrt{\frac{L}{C}},$$

$$r_2 = \sqrt{\frac{L}{C}},$$

$$K = \frac{\sqrt{(LC)}}{R}.$$

Case II. $RC > LS$, $S \neq 0$ (Fig. 94).

$$r_1 = 2\left(\sqrt{\frac{S}{R}}\right)\left(\frac{RL}{CR - LS}\right),$$

$$r_2 = \sqrt{\frac{R}{S}},$$

$$K = \frac{CR - LS}{2SR} \cdot \sqrt{\frac{S}{R}}.$$

Case III. $RC < LS$ (Fig. 95).

$$r_1 = 2\left(\sqrt{\frac{L}{C}}\right)\left(\frac{CR}{LS - CR}\right),$$

$$r_2 = \sqrt{\frac{L}{C}},$$

$$L_1 = \frac{2L\sqrt{(LC)}}{LS - CR}.$$

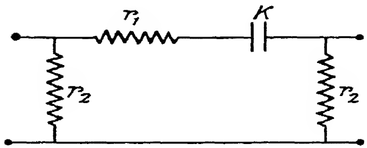


FIG. 94.

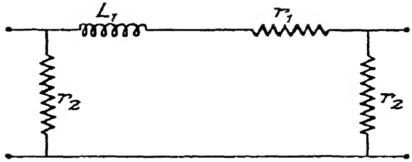


FIG. 95.

It will be found that Fig. 93 may be obtained from Fig. 90 by the method § 27 and so the two sections are exactly equivalent.

The T and Π sections of Figs. 91 and 94, however, are not equivalent for, though they have the same characteristic impedances, their propagation constants are different. Thus by the method of § 27 we can derive an additional Π section line balance from Fig. 91 and an additional T section from Fig. 94: similarly with Figs. 92 and 95.

§ 36. Line Balances. Bridge Sections.

Of these a very large number can be obtained by applying the results of Chapter IV. to any of the artificial lines obtained in §§ 34, 35.

Another series can be obtained from the artificial telephone line described in § 33, for if approximations to this are made in which $Z_0 \coth Pl/2$ and $Z_0 \tanh Pl/2$ are of reciprocal types and are both built out to the same degree of approximation, the characteristic impedance will be Z_0.

The simplest case of this series of artificial lines is the simple bridge section of Fig. 96, where

$$A = x(R + j\omega L)$$

$$B = \frac{1}{x} \cdot \frac{1}{(S + j\omega C)}$$

and x is any positive numerical quantity.

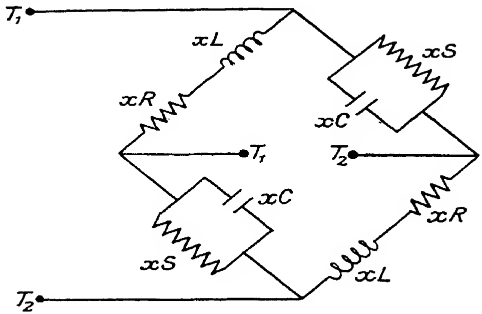


FIG. 96.

§ 37. Line Balances. Three-Element Artificial Lines.

Line balances can be obtained by using simple cases of the three-element artificial line described in Chapter II., § 18.

Examples are shown in Fig. 97 (a) and (b).

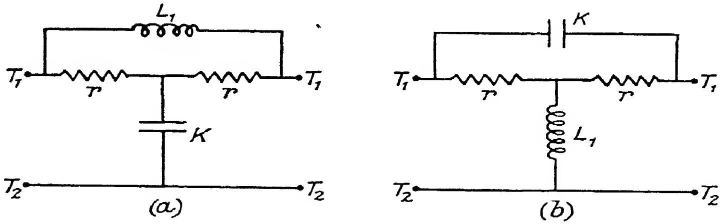


FIG. 97.

If, in Fig. 97(a) r, K, and L_1 are given, the values

$$r = \sqrt{\frac{L}{C}},$$

$$K = \frac{2\sqrt{CL}}{R},$$

$$L_1 = \frac{2\sqrt{CL}}{S},$$

it is found on inserting these values in 18.2 that

$$Z_0 = \sqrt{\frac{R + j\omega L}{S + j\omega C}}$$

and similarly with Fig. 97 (*b*), if

$$r = \sqrt{\frac{R}{S}},$$

$$K = \frac{C}{2\sqrt{RS}},$$

$$L_1 = \frac{L}{2\sqrt{RS}}.$$

The special case, where $D = 4B$, of Chapter II., § 18, leads to a general class of line balances if A and B are chosen so that

$$2AB = \frac{R + j\omega L}{S + j\omega C}.$$

Two simple cases are shown in Fig. 98 (*a*) and (*b*).

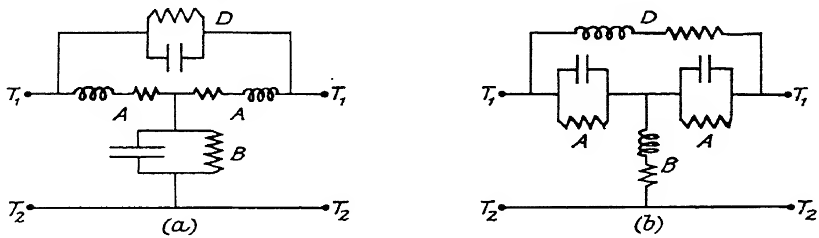


FIG. 98.

These three-element artificial lines have a characteristic impedance equal to

$$\sqrt{\frac{R + j\omega L}{S + j\omega C}},$$

provided that A, B, and D are given the values in Fig. 98(*a*):

$$A = \frac{x}{2}(R + j\omega L),$$

$$B = \frac{1}{x} \cdot \frac{1}{S + j\omega C},$$

$$D = 4B,$$

and in Fig. 98(*b*):

$$A = \frac{x}{2} \cdot \frac{1}{S + j\omega C},$$

$$B = \frac{1}{x}(R + j\omega L),$$

$$D = 4B,$$

where x is any numerical quantity.

The first of this pair may be compared with the simple artificial telephone line of § 32. It may be used as an artificial telephone line and is an improvement as the characteristic impedance can be made exactly equal to that of the uniform line.

It will also be found quite easy to devise a special case of the three-element artificial line of Fig. 42, § 18.

In the same way, four-element artificial lines can be obtained. Indeed, there appears to be no limit to the number of artificial lines that can be constructed to have the same characteristic impedance as a uniform line.

§ 38. Impedance Corrective Networks.

Consider first a uniform line having S = 0. We have seen from § 34 that the sending-end impedance of the infinite uniform line is equal to the sending-end impedance of the infinite T section artificial line of Fig. 99,

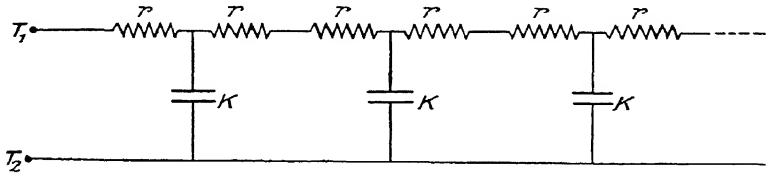


FIG. 99.

where

$$r = \sqrt{\frac{L}{C}},$$

$$K = \frac{2\sqrt{LC}}{R}.$$

This infinite network can be redrawn as in Fig. 100, where Y is an infinite artificial line composed of Π sections, of which one is shown in Fig. 101.

On referring back to Ex. 6 of Chapter III. it will be seen that if the network (Fig. 102) consisting of r, $Kr^2/2$ and r^2/Y in series is connected in parallel with the network of Fig. 100, the impedance of the combination is a pure resistance r.

On reciprocating Y with respect to r and adding the series

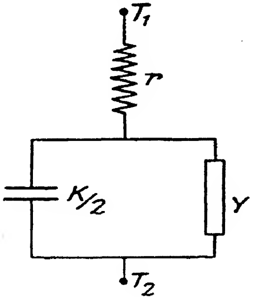


FIG. 100.

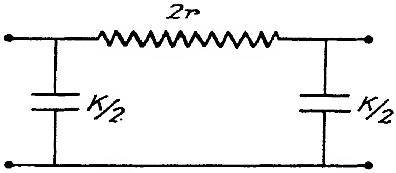


FIG. 101.

resistance r and inductance $Kr^2/2$, the network of Fig. 102 becomes the infinite network of Fig. 103, having the property that when connected in parallel with the infinite uniform line it reduces the impedance to a non-reactive resistance, r, equal in value to $\sqrt{\frac{L}{C}}$.

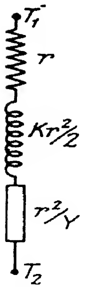


FIG. 102.

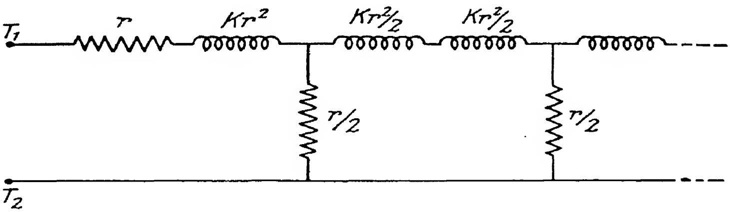


FIG. 103.

In a similar way, if $S \neq 0$ corrective networks can be obtained by applying the same methods to the artificial lines of Figs. 91 and 92.

From the corresponding Π sections of Figs. 93, 94 and 95 other corrective networks can be obtained in the same way or by using the result of Example 5, Chapter III.

§ 39. A Complete Distortion Correcting Artificial Line.

The question of attenuation and phase equalisation as such does not come within the scope of this book, but in this section we shall deal with a remarkable artificial line which has the property of equalising both attenuation and phase angle in a uniform line.

Already in § 33 we have dealt with an artificial line requiring a single infinity of simple impedance elements for its exact construction; the artificial line to be described requires, however, a double infinity of impedance elements. It is not suggested that it is a practical method but it is given rather as an example of the possibilities of artificial lines.

Distortion in telephonic transmission occurs owing to two causes: waves of different frequencies are in general attenuated in different degrees in passing along a telephone line; in addition their relative phase relations change. The only exception is the Heaviside Distortionless Line in which R, L, C and S, the resistance, inductance, capacity, and leakage respectively per unit length, are such that

$$\mathrm{RC} = \mathrm{LS}.$$

The question of attenuation and phase correction is of considerable importance; the ideal is to have both attenuation and phase shift independent of frequency.

Take the case of the uniform line without leakage: its propagation constant is $\alpha + j\beta$, where

$$\alpha + j\beta = \sqrt{(\mathrm{R} + j\omega\mathrm{L})j\omega\mathrm{C}},$$

$$\alpha = \sqrt{\tfrac{1}{2}\mathrm{C}\omega\{\sqrt{\mathrm{R}^2 + \omega^2\mathrm{L}^2} - \mathrm{L}\omega\}},$$

$$\beta = \sqrt{\tfrac{1}{2}\mathrm{C}\omega\{\sqrt{\mathrm{R}^2 + \omega^2\mathrm{L}^2} + \mathrm{L}\omega\}}.$$

α can be written as

$$\frac{\mathrm{CR}}{\sqrt{\mathrm{LC}}} - f_1(\omega),$$

and β as

$$\omega\sqrt{\mathrm{LC}} - f_2(\omega),$$

where $f_1(\omega)$ and $f_2(\omega)$ are functions of frequency.

The condition that a symmetrical transmission system shall be distortionless is that the propagation constant shall be of the form $\mathrm{X} + j\omega\mathrm{Y}$, where X and Y are independent of frequency.

Thus if an artificial line could be constructed having the same characteristic impedance $\mathrm{Z}_0 = \sqrt{\dfrac{\mathrm{R} + j\omega\mathrm{L}}{j\omega\mathrm{C}}}$ as the uniform line and

having a propagation constant $l\{f_1(\omega) + jf_2(\omega)\}$ and if this artificial line were inserted in a length l of the uniform line, the combination would have a characteristic impedance Z_0 and a combined propagation constant

$$l\left(\frac{CR}{\sqrt{LC}} + j\omega\sqrt{LC}\right)$$

and therefore it would propagate either a current or voltage wave without distortion if terminated by Z_0.

Putting $$f_1(\omega) + jf_2(\omega) = \psi,$$
a network such as Fig. 104 would constitute the required artificial line.

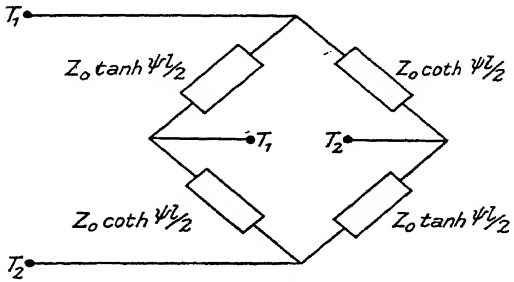


FIG. 104.

It therefore remains to be seen whether the impedances
$$Z_0 \tanh \psi l/2 \quad \text{and} \quad Z_0 \coth \psi l/2$$
can be put into physically realisable form.

It has already been shown in § 33 that, provided $Z_0\psi$ and Z_0/ψ can be put into physical form, then both $Z_0 \coth \psi l/2$ and $Z_0 \tanh \psi l/2$ can be realised physically in a number of ways, each of which employs a single infinity of numerical multiples of $Z_0\psi$ and Z_0/ψ. The problem therefore reduces to determining whether $Z_0\psi$ and Z_0/ψ can be physically realised.

Now

$$\psi = \frac{CR}{\sqrt{LC}} + j\omega\sqrt{LC} - \sqrt{(R + j\omega L)j\omega C},$$

$$Z_0\psi = \sqrt{\frac{R + j\omega L}{j\omega C}} \cdot \psi$$

$$= \sqrt{\frac{C}{L}} \cdot (R + j\omega L)\left\{\sqrt{\frac{R + j\omega L}{j\omega C}} - \sqrt{\frac{L}{C}}\right\}.$$

But from § 34 it follows that

$$\sqrt{\frac{R + j\omega L}{j\omega C}} - \sqrt{\frac{L}{C}}$$

is the impedance of the infinite network shown in Fig. 105, where all the resistances have the value $2\sqrt{\frac{L}{C}}$, and the capacities the value $\frac{2\sqrt{LC}}{R}$.

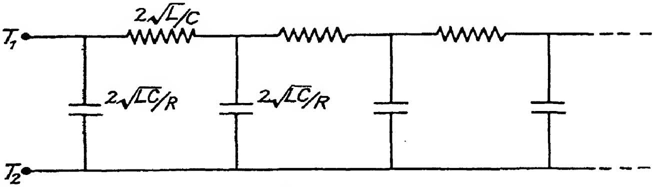


Fig. 105.

Multiplying all the impedances of this network by

$$\sqrt{\frac{C}{L}} \cdot (R + j\omega L)$$

we have $Z_0\psi$ as the impedance of the infinite network shown in Fig. 106.

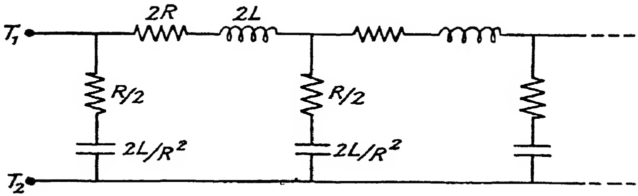


Fig. 106.

In a similar way Z_0/ψ is the impedance of the network shown in Fig. 107.

Thus both the impedances

$$Z_0 \coth \psi l/2 \quad \text{and} \quad Z_0 \tanh \psi l/2$$

can be physically realised. A double infinity of impedance elements, however, is required, but all the infinite networks are convergent and approximations can be made.

We have dealt only with the case in which $S = 0$, but if $S \neq 0$ corresponding networks can be devised in the same way.

The auxiliary network may be looked upon as a wave form converter, for suppose to produce a voltage form $F(t)$ at the far end of the uniform line it is necessary to apply a voltage form $F'(t)$ at the sending end, then the auxiliary network, if placed at the beginning of the uniform line, may be regarded as converting a given voltage form $F(t)$ into the required form, $F'(t)$.

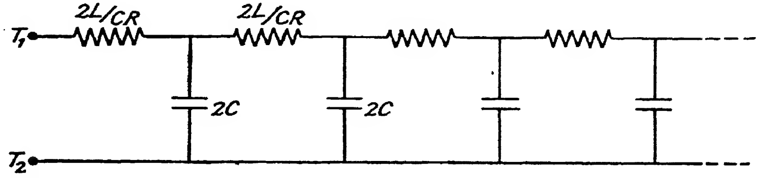


FIG. 107.

EXAMPLES AND NOTES.

1. The wireless aerial is often treated theoretically as a length of uniform transmission line open-circuited at the far end; its impedance is therefore

$$Z_0 \coth Pl.$$

Assuming that this treatment is justified then by means of the methods of § 33, three different infinite networks that are equivalent to the aerial can be obtained.

They are the reciprocals of Figs. 86, 88 and 89 with respect to Z_0, $l/2$ of course having to be replaced by l throughout.

2. Show that for the line balance section shown in Fig. 98(a)

$$\theta = Px\left(1 + \frac{P^2x^2}{12}\right) \text{ approx.}$$

CHAPTER VI.

THE COIL LOADED TELEPHONE CABLE.

§ 40. The Series Coil Loaded Cable.

The coil loaded cable consists of a uniform cable into which, at intervals of the order of one mile, series inductance coils are inserted. The suggestion of inserting such coils was made by Heaviside as a result of his classical work on telephonic transmission.

The subject of telephone loading as such does not come within the range of this book, but the determination of the constants of a loaded cable affords an excellent example of the use of artificial line theory.

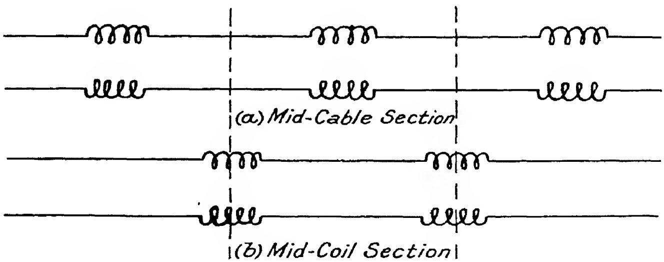


FIG. 108.

In order to facilitate the calculations it is necessary to consider the line as being composed of symmetrical sections. This can be done in two ways—the line can be considered as divided into sections by cuts either through the mid-points of consecutive lengths of cable, as in Fig. 108 (a), or through the mid-points of the loading coils as in Fig. 108 (b).

Either of these sections falls within our definition of an artificial line and can be treated in the usual way.

First consider the mid-cable section. Let R, L, S and C be the

constants per unit length of the cable itself, and let Z_0 and P be its characteristic impedance and propagation constant per unit length so that

$$Z_0 = \sqrt{\left(\frac{R + j\omega L}{S + j\omega C}\right)},$$
$$P = \sqrt{\{(R + j\omega L)(S + j\omega C)\}}.$$

Let l be the length of cable between loading coils, and let X be the added series impedance. Hence a mid-cable section consists of two lengths $l/2$ of cable with the series impedance X inserted between them. Each length of cable can be replaced by its equivalent T; further the section can be bisected, and a half-section, as shown in Fig. 109, obtained.

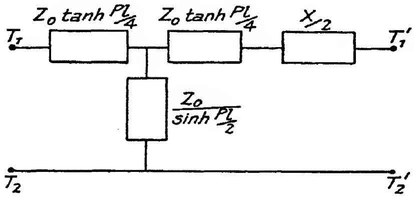


Fig. 109.

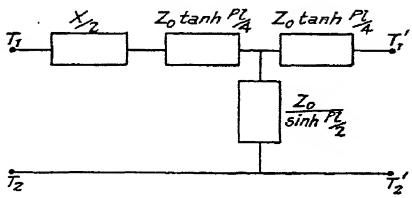


Fig. 110.

Let Z_0' and θ' be the constants of a single mid-cable section. Then $Z_0' \tanh \theta'/2$ is the impedance of the network of Fig. 109 short-circuited at the output, and either by working out this ladder network or by using ordinary telephone line transmission theory we have

$$Z_0' \tanh \theta'/2 = Z_0 \frac{\left(X \cosh \frac{Pl}{2}\right)\Big/ 2 + Z_0 \sinh \frac{Pl}{2}}{Z_0 \cosh \frac{Pl}{2} + \left(X \sinh \frac{Pl}{2}\right)\Big/ 2} \quad . \quad . \quad 40.1$$

$Z_0' \coth \theta'/2$ is the impedance of the network of Fig. 109 open-circuited at the output; it is therefore the impedance of a length $l/2$ of the uniform line open-circuited at the far end.

Thus

$$Z_0' \coth \theta'/2 = Z_0 \coth Pl/2 \quad . \quad . \quad . \quad 40.2$$

and therefore

$$Z_0' = Z_0 \sqrt{\left\{ \coth \frac{Pl}{2} \frac{\left(X \cosh \frac{Pl}{2}\right)\Big/2 + Z_0 \sinh \frac{Pl}{2}}{Z_0 \cosh \frac{Pl}{2} + \left(X \sinh \frac{Pl}{2}\right)\Big/2} \right\}}$$

$$= Z_0 \sqrt{\frac{Z_0 + \frac{X}{2} \coth \frac{Pl}{2}}{Z_0 + \frac{X}{2} \tanh \frac{Pl}{2}}} \quad . \quad . \quad . \quad . \quad . \quad 40.3$$

$\theta'/2$ can be obtained from 40.1 and 40.2; by a little manipulation it can be shown that

$$\cosh \theta' = \cosh Pl + \frac{X}{2Z_0} \sinh Pl. \quad . \quad . \quad 40.4$$

Turning now to the mid-coil or rather mid-load termination, we obtain a half-section as shown in Fig. 110.

If Z_0'' and θ'' are the constants of the section, we have

$$Z_0'' \coth \theta''/2 = X/2 + Z_0 \coth Pl/2, \quad . \quad . \quad 40.5$$

$$Z_0'' \tanh \theta''/2 = X/2 + Z_0 \tanh Pl/2, \quad . \quad . \quad 40.6$$

whence

$$Z_0'' = \sqrt{\{(Z_0 + \tfrac{1}{2}X \coth Pl/2)(Z_0 + \tfrac{1}{2}X \tanh Pl/2)\}} \quad . \quad 40.7$$

It can readily be shown that $\theta'' = \theta'$ and is therefore given by 40.4.

If the loading coils have a resistance r and an inductance H, it is only necessary to replace X by $r + j\omega H$.

The formulæ then become

$$Z_0' \text{ (mid-cable)} = Z_0 \sqrt{\frac{Z_0 + \frac{1}{2}(r + j\omega H) \coth Pl/2}{Z_0 + \frac{1}{2}(r + j\omega H) \tanh Pl/2}}$$

$$\cosh \theta' = \cosh Pl + \frac{r + j\omega H}{2Z_0} \sinh Pl.$$

$$Z_0'' \text{ (mid-coil)} = \sqrt{\{Z_0 + \tfrac{1}{2}(r + j\omega H) \coth Pl/2\}\{Z_0 + \tfrac{1}{2}(r + j\omega H) \tanh Pl/2\}}$$

$$\cosh \theta'' = \cosh \theta'.$$

It is interesting to compare this treatment with the original derivation of these formulæ from first principles by G. A. Campbell (*Phil. Mag.*, 1903, June) to see the great saving of labour due to the use of artificial line theory.

§ 41. The Shunt Loaded Cable.

Shunt loading has been suggested but does not appear to have been used in practice; nevertheless the necessary formulæ can be readily obtained.

Let there be a shunt load Y connected across the uniform line at distances l apart. As before we shall have two terminations, mid-load and mid-cable. The mid-cable section can be bisected, a half-section being shown in Fig. 111.

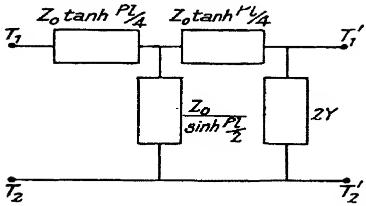


FIG. 111.

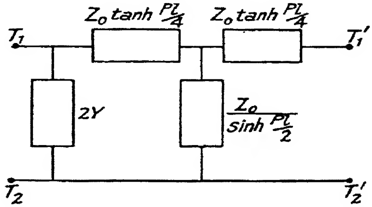


FIG. 112.

If Z_0' and θ' are the mid-cable constants,

$$Z_0' \tanh \theta'/2 = Z_0 \tanh Pl/2$$

and

$$Z_0' \coth \theta'/2 = Z_0 \frac{2Y \cosh Pl/2 + Z_0 \sinh Pl/2}{Z_0 \cosh Pl/2 + 2Y \sinh Pl/2} .$$

$$\therefore \quad Z_0' = Z_0 \sqrt{\left(\frac{2Y + Z_0 \tanh Pl/2}{2Y + Z_0 \coth Pl/2}\right)}, \qquad . \qquad . \quad 41.1$$

while θ' can be shown to be given by

$$\cosh \theta' = \cosh Pl + Z_0/2Y \sinh Pl/2. \quad . \quad . \quad 41.2$$

For the mid-load case a half-section is shown in Fig. 112.

If Z_0'' and θ'' are the constants, we see that $Z_0'' \tanh \theta''/2$ is the impedance of $2Y$ and $Z_0 \tanh Pl/2$ connected in parallel, while $Z_0'' \coth \theta''/2$ is the impedance of $2Y$ and $Z_0 \coth \mathrm{Pl}/2$ in parallel.

Thus

$$Z_0'' \tanh \theta''/2 = \frac{2YZ_0 \tanh Pl/2}{2Y + Z_0 \tanh Pl/2}$$

and

$$Z_0'' \coth \theta''/2 = \frac{2YZ_0 \coth Pl/2}{2Y + Z_0 \coth Pl/2} .$$

Hence

$$Z_0'' = \sqrt{\left\{\frac{YZ_0}{(Y + \frac{1}{2}Z_0 \tanh Pl/2)(Y + \frac{1}{2}Z_0 \coth Pl/2)}\right\}}. \qquad 41.3$$

EXAMPLES.

Series Loaded Cable.

1. Show that

$$\frac{Z_0}{\sinh Pl} = \frac{Z_0''}{\sinh \theta''}$$

$$X/2 + Z_0 \tanh Pl/2 = Z_0'' \tanh \theta''/2$$

$$Z_0'' \coth \theta'' = X/2 + Z_0 \coth Pl.$$

2. Putting (T, x) for $\sqrt{(Z_0 + \frac{1}{2}X \tanh Px)}$ and (C, x) for

$$\sqrt{(Z_0 + \frac{1}{2}X \coth Px)},$$

show that the impedance of an infinite loaded cable measured at a length l' in the cable from the first load impedance (l being the distance between loads) is

$$Z_0 \frac{(C, l/2)(T, l/2) + (C, l')^2 \tanh Pl'}{(C, l/2)(T, l/2) \tanh Pl' + (T, l')^2},$$

and by putting $l' = l/2$, obtain the formula for Z_0'.

3. With the same notation as Example 2,

$$Z_0' = Z_0 \frac{(C, l/2)}{(T, l/2)},$$

$$Z_0'' = (T, l/2)(C, l/2).$$

4. If $X/2Z_0 = \tanh \gamma$,

$$Z_0' = Z_0 \tanh (Pl/2 + \gamma) \coth Pl/2.$$

CHAPTER VII.

FILTERS.

§ 42. Filters in General.

We shall define a filter as an artificial line of one or more sections composed entirely of pure reactances, i.e. of loss-free self and mutual inductances and condensers. Such a filter is, of course, not realisable in practice, for there must be losses in both inductances and condensers. The assumption of freedom from loss, however, simplifies the treatment enormously and is, in general, a good approximation to the truth.

Consider any filter section; if Z_0 and θ are its constants, then at any one frequency $Z_0 \tanh \theta$, the impedance of a single section short-circuited at the output must be imaginary since it is the impedance of a network composed entirely of reactances.

Similarly $Z_0 \coth \theta$ is wholly imaginary.

Consider first the value of Z_0^2; it is the product of two entirely imaginary quantities, viz. $Z_0 \tanh \theta$ and $Z_0 \coth \theta$, and is therefore real at all frequencies. Two cases, however, arise; for some ranges of frequency it may be positive, for others negative. Thus Z_0 for some frequencies may be real, and for others imaginary.

Turn now to the consideration of θ, the propagation constant per section. We have seen that in the case of the filter section $Z_0 \tanh \theta$ is imaginary; hence for frequencies at which Z_0 is real $\tanh \theta$ must be imaginary.

Now
$$\theta = \alpha + j\beta,$$
and
$$\tanh \theta = \tanh (\alpha + j\beta) = \frac{\sinh 2\alpha + j \sin 2\beta}{\cosh 2\alpha + \cos 2\beta}.$$

In order that $\tanh \theta$ may be wholly imaginary $\sinh 2\alpha$ must vanish, i.e. we must have $\alpha = 0$.

Thus for frequencies at which Z_0 is real the attenuation constant is zero; at such frequencies in an infinite filter line or in a filter

consisting of a finite number of sections terminated by a resistance equal to Z_0, the current and voltage across the terminals of any section and across the terminal load are the same in amplitude as at the input terminals; the phases, however, will differ by the angle β per section.

Again, in the case where Z_0 is imaginary, $\tanh\theta$ must be wholly real. This obtains only if $\sin 2\beta$ vanishes and α is not zero.

We must therefore have $\beta = m\pi/2$, where m is any integer. But it is possible to go further—in most of the filters we shall deal with $Z_0 \tanh\theta/2$ and $Z_0 \coth\theta/2$ are physical pure reactive networks; if we took an odd value of m then $\tanh\theta/2$ would be of the form $\tanh(\alpha/2 + jm\pi/4)$ and on expanding this it will be partly real and partly imaginary and thus $Z_0 \tanh\theta/2$ would not be entirely imaginary. Thus we must reject odd multiples of $\pi/2$, and therefore when Z_0 is imaginary β must be of the form $m\pi$ where m is any integer.

The cases where $Z_0 \tanh\theta/2$ and $Z_0 \coth\theta/2$ are physically realisable networks are the Bridge Section and any section to which the Bisection theorem applies, but it has been shown by Cauer ("Preuss. Akad. Wiss. Berlin," Ber. 33, 1927) that for any filter section $Z_0 \tanh\theta/2$ and $Z_0 \coth\theta/2$, even if not physically realisable, must have the properties of a purely reactive network and thus be purely imaginary.

Thus for any filter section when Z_0^2 is negative β must be of the form $m\pi$, where m is any integer.

Recapitulating there are two cases as below:—

$$\text{Case I.}\quad \begin{cases} Z_0^2 \text{ is positive,} \\ \alpha \text{ is zero,} \\ \beta \text{ is finite.} \end{cases}$$

$$\text{Case II.}\quad \begin{cases} Z_0^2 \text{ is negative,} \\ \alpha \text{ is finite,} \\ \beta \text{ is } m\pi\text{, where } m\text{, is an integer.} \end{cases}$$

A frequency for which the first conditions hold is called a transmitting or non-attenuating frequency of the filter—a frequency for which the second conditions hold, an attenuating frequency,

It will be seen from previous chapters that filters of innumerable types and of endless complexity can be constructed. The remainder of this chapter will be devoted to describing some of the simpler types, together with the general transmission properties of

filters. For the more specialised design of complicated types of filter for particular technical purposes, the papers by Zobel in the "Bell System Technical Journal," Vol. II., 1923, may be consulted; also "Les Filtres Électriques," by P. David (Paris: Gauthier-Villars, 1926).

§ 43. Simple T Section Low-pass Filter.

Consider the simple filter section of Fig. 113.

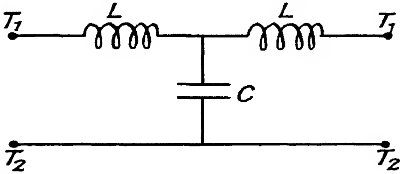


FIG. 113.

Its characteristic impedance is given by

$$\begin{aligned} Z_0^2 &= (j\omega L)^2 + 2j\omega L \cdot \frac{1}{j\omega C} \\ &= -\omega^2 L^2 + 2\frac{L}{C} \\ &= \frac{L}{C}(2 - \omega^2 LC). \qquad 43.1 \end{aligned}$$

For values of ω less than

$$\omega_0 = \sqrt{\frac{2}{LC}} \qquad 43.2$$

Z_0^2 is positive; if $\omega > \omega_0$, Z_0^2 is negative.

Thus the attenuation constant is zero for all values of ω lying between $\omega = 0$ and $\omega = \omega_0$; this range we shall term a *transmitting* or *non-attenuating frequency band.*

Similarly the range of ω from $\omega = \omega_0$ to $\omega = \infty$ is an *attenuating frequency band.*

We can determine the values of α and β by considering $Z_0 \tanh\theta$, $\cosh\theta$ or $Z_0 \tanh\theta/2$.

First take the simple formula

$$\begin{aligned} \cosh\theta &= 1 + A/B \\ &= 1 - \omega^2 LC \\ &= 1 - 2\frac{\omega^2}{\omega_0^2}. \qquad 43.3 \end{aligned}$$

For the transmitting band $\omega = 0$ to $\omega = \omega_0$ we have $\alpha = 0$, $\theta = j\beta$, so that equation 43.3 becomes

$$\cosh j\beta = 1 - 2\frac{\omega^2}{\omega_0^2},$$

i.e.

$$\cos \beta = 1 - 2\frac{\omega^2}{\omega_0^2}. \qquad 43.4$$

For small values of β

$$\cos \beta = 1 - \beta^2/2,$$

and thus comparing with 43.4 we see that for ω/ω_0 small

$$\beta = 2\frac{\omega}{\omega_0} \text{ approximately}. \qquad 43.5$$

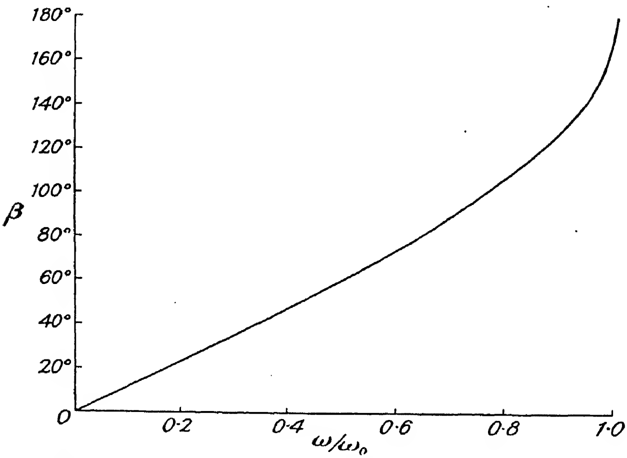


Fig. 114.—Phase angle β for low-pass T filter throughout the transmitting band.

The curve plotted in Fig. 114 shows the change in β throughout the non-attenuating range of the low-pass T section filter.

In the attenuating band either

$$\cosh \alpha = 1 - 2\frac{\omega^2}{\omega_0^2},$$

or

$$\cosh (\alpha + j\pi) = 1 - 2\frac{\omega^2}{\omega_0^2},$$

according as $\beta = 0$ or π. But since $\omega > \omega_0$ the right side must be negative and less than -1; moreover, $\cosh \alpha$ is essentially

positive, hence we must take β as π, in which case

$$\cosh\theta = \cosh(\alpha + j\pi) = 1 - 2\frac{\omega^2}{\omega_0^2},$$

i.e.

$$\cosh\alpha = 2\frac{\omega^2}{\omega_0^2} - 1, \qquad 43.6$$

since

$$\cosh(\alpha + j\pi) = -\cosh\alpha.$$

Values of α for various values of ω/ω_0 are shown in Fig. 115.

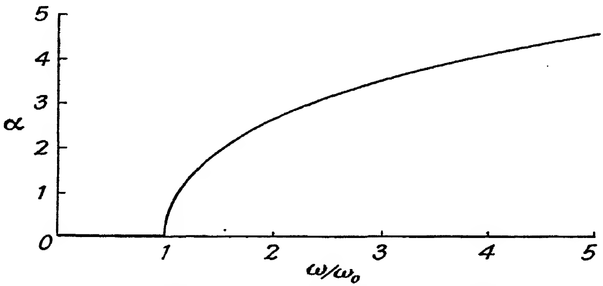


Fig. 115.—Attenuation Constant for low-pass T filter.

§ 44. Simple T Section High-pass Filter.

The simple T high-pass filter is shown in Fig. 116.

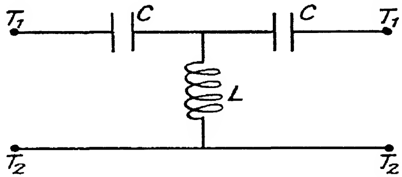


Fig. 116.

The characteristic impedance is given by

$$\begin{aligned} Z_0^2 &= \left(\frac{1}{j\omega C}\right)^2 + 2\frac{1}{j\omega C}\cdot j\omega L \\ &= \frac{2L}{C} - \frac{1}{\omega^2 C^2} \\ &= \frac{2L}{C}\left(1 - \frac{1}{2\omega^2 LC}\right) \\ &= \frac{2L}{C}\left(1 - \frac{\omega_0^2}{\omega^2}\right), \qquad 44.1 \end{aligned}$$

where
$$\omega_0 = \frac{1}{\sqrt{2LC}} \quad . \quad . \quad . \quad . \quad . \qquad 44.2$$

Thus, for frequencies given by $0 \leqq \omega \leqq \omega_0$, Z_0^2 is negative, so that this range is an attenuating band. The range for $\omega > \omega_0$ is a transmitting band, since Z_0^2 is positive.

Determination of α and β :—

$$\begin{aligned}\cosh\theta &= 1 + A/B \\ &= 1 + \frac{1}{j\omega C}\Big/ j\omega L \\ &= 1 - \frac{1}{\omega^2 LC} \\ &= 1 - 2\frac{\omega_0^2}{\omega^2}.\end{aligned}$$

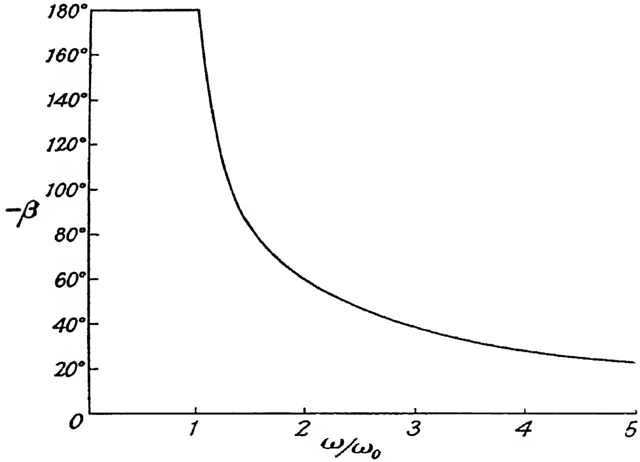


Fig. 117.—Phase angle for high-pass T filter.

For the non-attenuating band, where $\omega > \omega_0$,

$$\cosh j\beta = 1 - 2\frac{\omega_0^2}{\omega^2},$$

i.e.
$$\cos\beta = 1 - 2\frac{\omega_0^2}{\omega^2}.$$

For high values of ω, such that ω_0/ω is small,

$$\beta = \pm 2\frac{\omega_0}{\omega} \text{ approximately.}$$

The curve given in Fig. 117 shows the values of β over the complete frequency range. The determination of β from a cosine formula leaves an ambiguity of sign, but this can be removed by taking into consideration the fact that $jZ_0 \tan \beta/2$ is $-j/C\omega$ and so β must be negative.

In the attenuating band,

$$\cosh(\alpha - j\pi) = 1 - 2\frac{\omega_0^2}{\omega^2},$$

i.e.

$$\cosh \alpha = 2\frac{\omega_0^2}{\omega^2} - 1.$$

Values of α for various values of ω/ω_0 are plotted in Fig. 118.

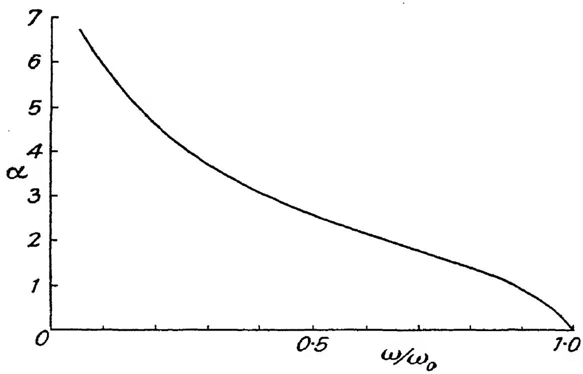


Fig. 118.—Attenuation Constant for high-pass T filter.

§ 45. General Transmission Formulæ for a Filter—Transmitting Band.

The transmission formulæ of a filter, for frequencies within a transmitting band, are the general equations of the artificial line modified by putting $\alpha = 0$; thus

$$\cosh n\theta = \cosh jn\beta = \cos n\beta,$$
$$\sinh n\theta = \sinh jn\beta = j \sin n\beta.$$

The sending-end impedance of an n section filter terminated by an impedance z is

$$z_n = Z_0 \frac{z \cos n\beta + jZ_0 \sin n\beta}{Z_0 \cos n\beta + jz \sin n\beta}, \qquad 45.1$$

while the receiving-end impedance is

$$y_n = jZ_0 \sin n\beta + z \cos n\beta \quad . \quad . \quad . \quad 45.2$$

where it is known also that Z_0 is real.

Thus the complex hyperbolic functions disappear, to be replaced by simple circular functions.

We see at once that if $z = Z_0$, then z_n is equal to Z_0, while

$$y_n = Z_0 (\cos n\beta + j \sin n\beta);$$

that is, the input and output currents are equal in magnitude but differ in phase. The same remark applies to the input and output voltages.

§ 46. General Transmission Formulæ for a Filter—Attenuating Band.

The general artificial line formulæ have to be modified by putting

$$\theta = a$$
$$\text{or } \theta = a \pm j\pi.$$

If $\theta = a$ we have

$$\cosh n\theta = \cosh na,$$
$$\sinh n\theta = \sinh na,$$

while if $\theta = a \pm j\pi$ we have

$$\begin{aligned}\cosh n\theta &= \cosh (na \pm jn\pi)\\ &= (-1)^n \cosh na,\\ \sinh n\theta &= \sinh (na \pm jn\pi)\\ &= (-1)^n \sinh na.\end{aligned}$$

Hence

$$z_n = Z_0 \cdot \frac{z \cosh na + Z_0 \sinh na}{Z_0 \cosh na + z \sinh na} \quad . \quad . \quad 46.1$$

and

$$y_n = \pm (Z_0 \sinh na + z \cosh na), \quad . \quad . \quad 46.2$$

the minus sign being taken when $\beta = \pi$ and n is odd. Z_0 is, of course, imaginary.

We can at once verify the attenuation by putting $z = Z_0$, in which case

$$\begin{aligned}y_n &= \pm Z_0 (\cosh na + \sinh na)\\ &= \pm Z_0 e^{na}.\end{aligned}$$

Hence, if $z = Z_0$ the output current is equal to the input current diminished by e^{na}. If the attenuation is considerable $\cosh na$ and

sinh na both approximate to $\frac{1}{2}e^{na}$, so that corresponding to 46.1 and 46.2 we have approximately

$$\left.\begin{aligned} z_n &= Z_0 \\ y_n &= \pm \tfrac{1}{2}e^{na}(Z_0 + z) \end{aligned}\right\} \qquad . \quad . \quad . \quad 46.3$$

§ 47. Filters having Two Transmitting Bands.

Consider a T section filter of which a half section is shown in Fig. 119 (a) and the complete section in Fig. 119 (b).

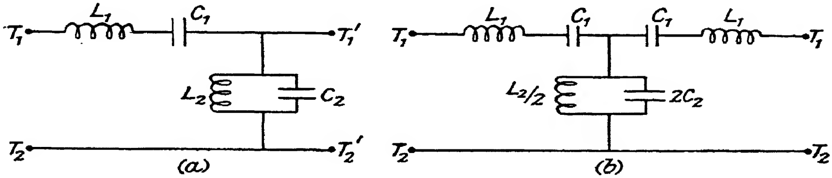


FIG. 119.

We have

$$Z_0 \tanh \theta/2 = j\omega L_1 + \frac{1}{j\omega C_1}$$

$$= \frac{1}{j\omega C_1}(1 - \omega^2 L_1 C_1); \qquad . \quad . \quad . \quad 47.1$$

$$Z_0 \coth \theta/2 = j\omega L_1 + \frac{1}{j\omega C_1} + \frac{j\omega L_2/j\omega C_2}{j\omega L_2 + \dfrac{1}{j\omega C_2}}$$

$$= \frac{1}{j\omega C_1}(1 - \omega^2 L_1 C_1) + \frac{j\omega L_2}{1 - \omega^2 L_2 C_2}. \qquad 47.2$$

Substituting

$$\left.\begin{aligned} \frac{1}{L_1 C_1} &= \omega_1^2 \\ \frac{1}{L_2 C_2} &= \omega_2^2 \\ C_1 &= kC_2 \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad 47.3$$

$Z_0 \coth \theta/2$ reduces readily to

$$\frac{1}{j\omega C_1} \cdot \frac{(\omega_1^2 - \omega^2)(\omega_2^2 - \omega^2) - k\omega^2\omega_1^2}{\omega_1^2(\omega_2^2 - \omega^2)},$$

which can be put in the form

$$\frac{1}{j\omega C_1} \frac{(\omega^2 - \omega_3^2)(\omega^2 - \omega_4^2)}{\omega_1^2(\omega_2^2 - \omega^2)},$$

where
$$\left.\begin{aligned} \omega_3{}^2\omega_4{}^2 &= \omega_1{}^2\omega_2{}^2 \\ \omega_3{}^2 + \omega_4{}^2 &= \omega_2{}^2 + \omega_1{}^2(1 + k) \\ \omega_3 &< \omega_4 \end{aligned}\right\} \quad . \quad . \quad 47.4$$

The characteristic impedance can now be expressed as

$$Z_0{}^2 = -\frac{1}{\omega^2 C_1{}^2}\cdot\frac{(\omega^2 - \omega_1{}^2)(\omega^2 - \omega_3{}^2)(\omega^2 - \omega_4{}^2)}{\omega_1{}^4(\omega^2 - \omega_2{}^2)} \quad . \quad . \quad 47.5$$

Consider the relations among ω_1, ω_2, ω_3 and ω_4; since $\omega_3\omega_4 = \omega_1\omega_2$, ω_3 and ω_4 must lie either both inside or both outside the range $\omega_1 \leqq \omega \leqq \omega_2$: but further,

$$\begin{aligned} (\omega_3 - \omega_4)^2 &= \omega_3{}^2 + \omega_4{}^2 - 2\omega_3\omega_4 \\ &= \omega_2{}^2 + \omega_1{}^2(1 + k) - 2\omega_1\omega_2 \\ &= (\omega_2 - \omega_1)^2 + k\omega_1{}^2, \end{aligned}$$

hence $\omega_3 - \omega_4 > \omega_2 - \omega_1$, and therefore ω_3 and ω_4 must lie outside the range $\omega_1 \leqq \omega \leqq \omega_2$.

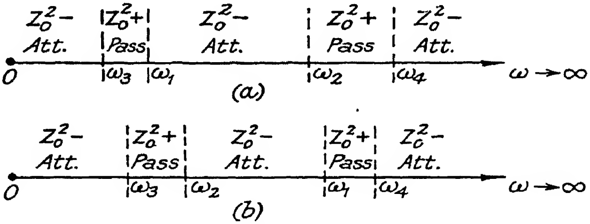


Fig. 120.

There are two cases to be considered, viz. $\omega_1 < \omega_2$ and $\omega_1 > \omega_2$. These are shown graphically in Fig. 120.

Taking the first case, suppose ω lies between 0 and ω_3; then $(\omega^2 - \omega_1{}^2)$, $(\omega^2 - \omega_2{}^2)$, $(\omega^2 - \omega_3{}^2)$, $(\omega^2 - \omega_4{}^2)$ will each be negative; hence $Z_0{}^2$ is negative and therefore 0 to ω_3 is an attenuating band. If ω lies between ω_3 and ω_1, $(\omega^2 - \omega_1{}^2)$, $(\omega^2 - \omega_2{}^2)$, $(\omega^2 - \omega_4{}^2)$ will each be negative, but $(\omega^2 - \omega_3{}^2)$ will be positive; thus $Z_0{}^2$ is positive and therefore from ω_3 to ω_1 is a transmitting band.

Similarly, ω_1 to ω_2 and ω_4 to ∞ are attenuating bands, and ω_2 to ω_4 is a transmitting band.

For the second case, $\omega_1 > \omega_2$, the results are shown in Fig. 120(*b*).

This filter has therefore two distinct passing bands. In practice a single passing band is usually required; this can be obtained from the above in various ways.

First, put $L_1 = 0$, then $\omega_1^2 = \infty$, so that ω_1 must be greater than ω_2 and we are dealing with the case shown in Fig. 120(*b*). One band has been displaced to infinity and we are left with a single band filter passing from ω_3 to ω_2. To find ω_3 let L_1 become very small; then ω_4 and ω_1 become so large that ω_3 and ω_2 can be neglected in the second equation of 47.4.

Hence ultimately, as $\omega_1 \to \infty$,

$$\frac{\omega_4^2}{\omega_1^2} \to (1 + k).$$

Substituting this in the first equation of 47.4,

$$\frac{\omega_3}{\omega_2} = \frac{\omega_1}{\omega_4} = \frac{1}{\sqrt{1 + k}}.$$

Hence $$\omega_3 = \frac{\omega_2}{\sqrt{1 + k}} = \frac{1}{\sqrt{L_2(C_1 + C_2)}} \quad . \quad . \quad . \quad 47.6$$

The filter now passes all frequencies in the range

$$\frac{1}{\sqrt{L_2(C_1 + C_2)}} < \omega < \frac{1}{\sqrt{L_2 C_2}}$$

and attenuates all other frequencies.

Suppose now, in addition to making $L_1 = 0$, we put $C_2 = 0$, then ω_3 becomes $1/\sqrt{L_2 C_1}$ and $\omega_2 = \infty$ and we have returned to the simple high-pass filter of § 44.

A single band filter can be obtained in another way, viz. by making the two bands join into one by putting $\omega_2 = \omega_1$, i.e. $L_1 C_1 = L_2 C_2$, then from equations 47.4

$$\begin{aligned} \omega_3 \omega_4 &= \omega_1^2, \\ \omega_3^2 + \omega_4^2 &= \omega_1^2(2 + k), \\ (\omega_4 - \omega_3)^2 &= k\omega_1^2, \\ (\omega_4 + \omega_3)^2 &= \omega_1^2(4 + k). \end{aligned}$$

$$\therefore \omega_4 = \omega_1 . \frac{\sqrt{k} + \sqrt{(4 + k)}}{2}, \quad . \quad . \quad . \quad 47.7$$

$$\omega_3 = \omega_1 . \frac{\sqrt{(4 + k)} - \sqrt{k}}{2} \quad . \quad . \quad . \quad 47.8$$

The filter now has a single transmission band from ω_3 to ω_4.

The Π section band filter of Fig. 121 can be treated in an exactly similar way.

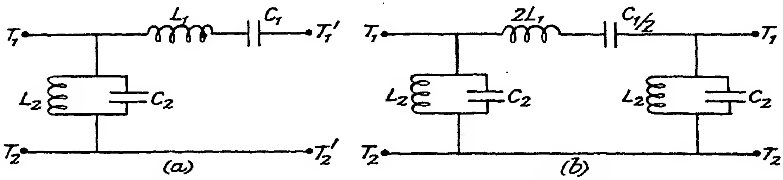
These double and single band filters were first described by G. A. Campbell in U.S. Patents, Nos. 1,227,113 and 1,227,114.

§ 48. Filter Terminated by a Resistance. Variable Number of Sections—Transmitting Band.

The sending-end impedance is

$$z_n = Z_0 \frac{r \cos n\beta + jZ_0 \sin n\beta}{Z_0 \cos n\beta + jr \sin n\beta}, \quad . \quad . \quad 48.1$$

where Z_0 is real and r is the value of the terminating resistance.

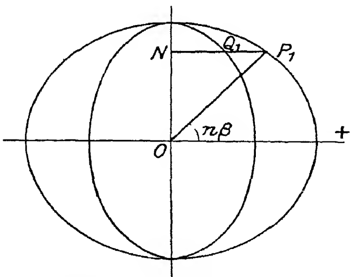


Half section. Full section.

FIG. 121.

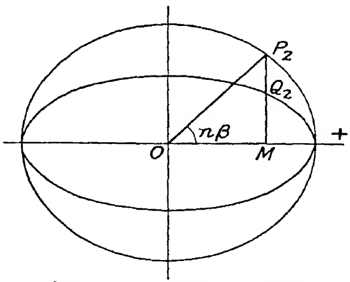
Suppose that n, the number of sections, is varied, or that β is varied by varying the type of filter section, Z_0 being kept constant. We shall consider first how the factor $r \cos n\beta + jZ_0 \sin n\beta$ varies with n. Suppose that $r < Z_0$.

Taking real and imaginary axes with O as origin, draw a circle of radius Z_0 as shown in Fig. 122; construct an ellipse, centre O,



$Q_1 = r \cos n\beta + jZ_0 \sin n\beta$

FIG. 122.



$Q_2 = Z_0 \cos n\beta + j r \sin n\beta$

FIG. 123.

with major axis $2Z_0$ and minor axis $2r$, the major axis lying along the imaginary axis. Next draw a line OP_1, making an angle $n\beta$ with the positive real axis, cutting the circle in P_1; draw P_1N perpendicular to the imaginary axis, cutting the ellipse in Q_1.

From a simple property of the ellipse we have

$$\frac{Q_1N}{P_1N} = \frac{r}{Z_0}.$$

Hence Q_1 is the point

$$r \cos n\beta + jZ_0 \sin n\beta.$$

Similarly, to obtain $Z_0 \cos n\beta + jr \sin n\beta$ draw a circle, centre O and radius Z_0, as in Fig. 123; draw an ellipse, centre O, axes $2Z_0$ and $2r$, the major axis $2Z_0$ lying along the real axis. Draw a line OP_2, making an angle $n\beta$ with the positive real axis, cutting the circle in P_2; draw P_2M perpendicular to the real axis, cutting the ellipse in Q_2.

Then Q_2 is the point

$$Z_0 \cos n\beta + jr \sin n\beta.$$

We see at once that both

$$|Z_0 \cos n\beta + jr \sin n\beta|$$
$$\text{and} \quad |r \cos n\beta + jZ_0 \sin n\beta|$$

must lie between Z_0 and r.

Hence $|z_n|$ must lie between

$$Z_0 . \frac{Z_0}{r} \quad \text{and} \quad Z_0 . \frac{r}{Z_0},$$

i.e. $$|z_n| \text{ lies between } \frac{Z_0^2}{r} \text{ and } r.$$

Next consider the angle of z_n; let it be ψ. Then

$$\psi = \tan^{-1}\left(\frac{Z_0 \sin n\beta}{r \cos n\beta}\right) - \tan^{-1}\left(\frac{r \sin n\beta}{Z_0 \cos n\beta}\right);$$

whence $$\tan \psi = \frac{Z_0^2 - r^2}{2Z_0 r} \sin 2n\beta. \qquad . \qquad . \qquad 48.2$$

Hence ψ must lie between

$$\pm \tan^{-1} \frac{Z_0^2 - r^2}{2Z_0 r}$$

and is zero if $r = Z_0$.

The receiving-end impedance

$$y_n = r \cos n\beta + jZ_0 \sin n\beta \quad . \qquad . \qquad . \qquad 48.3$$

does not require much notice, for an examination of Fig. 122 shows that Q_1 represents the receiving-end impedance, and therefore y_n traverses an ellipse as n is varied; $|y_n|$ lies between Z_0 and r.

The ratio of the receiving-end voltage to the sending-end voltage is

$$\frac{V_n}{V_0} = \frac{r}{r \cos n\beta + jZ_0 \sin n\beta} \cdot \qquad 48.4$$

and therefore

$$\left|\frac{V_n}{V_0}\right| \text{ lies between 1 and } \frac{r}{Z_0}.$$

§ 49. Filter Terminated by a Resistance. Variation of V_n/V_0 and z_n with Frequency and Number of Sections—Transmitting Band.

Consider a simple Π section filter in which A is an inductance L and B a capacity C. The characteristic impedance is given by

$$Z_0^2 = \frac{L}{2C} \cdot \frac{1}{1 - \omega^2/\omega_0^2},$$

where

$$\omega_0^2 = \frac{2}{LC}.$$

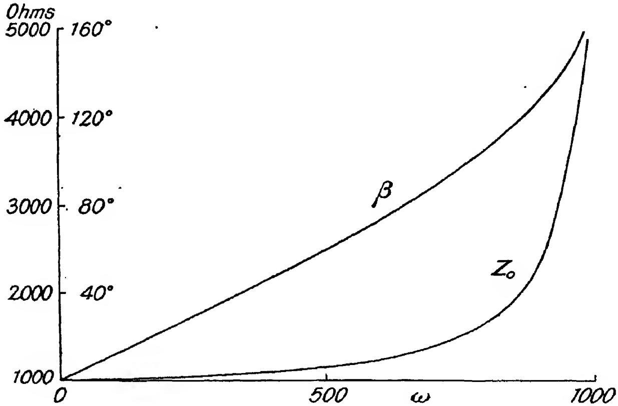


FIG. 124.

We see that Z_0^2 is positive if $\omega < \omega_0$ and is negative if $\omega > \omega_0$. Hence the range from $\omega = 0$ to $\omega = \omega_0$ is a transmitting band, and the range from $\omega = \omega_0$ to $\omega = \infty$ is an attenuating band.

We shall now examine the action of such a filter in the transmitting band. As a concrete example let us find the values of L and C such that

$$\omega_0 = 1000 \text{ rad./sec.}$$
$$Z_0 = 1000 \text{ ohms when } \omega = 0$$

and let the terminating resistance be 1500 ohms.

Then L and C are given by

$$10^6 = \frac{L}{2C}$$

and
$$10^6 = \frac{2}{LC},$$

whence
$$L = 2 \text{ henries}$$
$$C = 1 \text{ microfarad.}$$

In Fig. 124 are plotted Z_0 and β for values of ω up to the cut off $\omega_0 = 1000$.

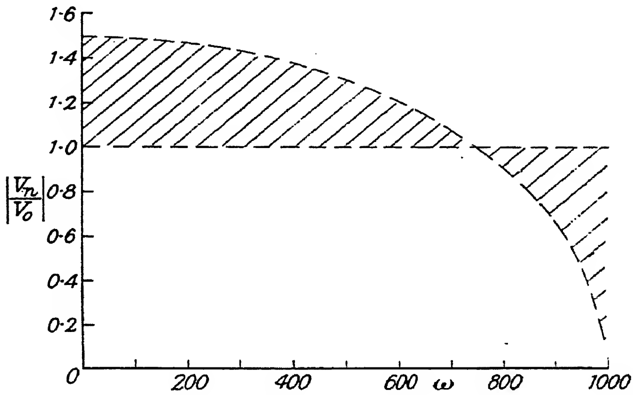


FIG. 125.

If V_0 is the input voltage and V_n the voltage across the terminating resistance, then, as we have seen in the previous section, $|V_n/V_0|$ must lie between 1 and r/Z_0. Hence if we draw the unity line and the curve of r/Z_0 against ω, as in Fig. 125, $|V_n/V_0|$ must lie in the area, shown shaded, between these limits.

We now see that for values of ω up to $\omega = 745$, V_n cannot be less than V_0; while for $\omega > 745$, V_n cannot be greater than V_0.

From 48.4 V_n/V_0 will have the value ± 1 for $\omega = 0$, for then $\beta = 0$, and for every value of ω for which $\sin n\beta = 0$, i.e. whenever $n\beta$ is equal to $m\pi$ where m is any integer. Similarly for every value of ω corresponding $\cos n\beta = 0$, i.e. whenever $n\beta$ is of the form $\pi/2 + m\pi$, where m is any integer, V_n/V_0 will be equal $\pm r/jZ_0$.

When $\beta = 180°$ and $\omega = \omega_0$ we have to consider the equation more closely; $r \cos n\beta$ becomes $(-1)^n r$ but $Z_0 \sin n\beta$ takes the form $\infty \times 0$ and it is necessary to determine the limit.

We have

$$\cos \beta = 1 - \omega^2 LC$$
$$= 1 - 2\frac{\omega^2}{\omega_0^2},$$

which can be written

$$-\cos \beta = 1 - 2(1 - \omega^2/\omega_0^2),$$

or

$$\cos(\pi - \beta) = 1 - 2(1 - \omega^2/\omega_0^2).$$

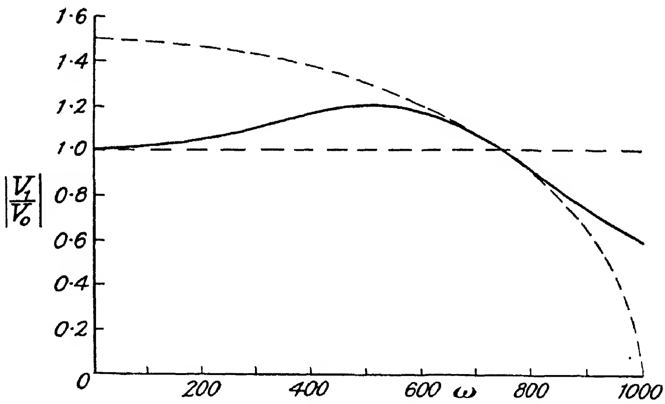


FIG. 126.

As ω approaches ω_0, $(\pi - \beta)$ diminishes and $(1 - \omega^2/\omega_0^2)$ becomes small; but when $(\pi - \beta)$ is small

$$\cos(\pi - \beta) = 1 - \frac{(\pi - \beta)^2}{2\,!} \text{ very approximately.}$$

Thus as ω approaches ω_0, $(\pi - \beta)$ tends to the value $2\sqrt{(1 - \omega^2/\omega_0^2)}$. Hence when $(\pi - \beta)$ is small

$$\sin n\beta = (-1)^{n-1}\, 2n\sqrt{(1 - \omega^2/\omega_0^2)}, \text{ approx.}$$

Since

$$Z_0 = \sqrt{\frac{L}{2C}} \cdot \frac{1}{\sqrt{1 - \omega^2/\omega_0^2}},$$

$Z_0 \sin n\beta$ will approach the value

$$(-1)^{n-1}\sqrt{\frac{L}{2C}} \cdot \frac{1}{\surd(1-\omega^2/\omega_0^2)} \times 2n\surd(1-\omega^2/\omega_0^2)$$
$$= (-1)^{n-1}\, 2n\sqrt{\frac{L}{2C}}.$$

Thus for $\omega = \omega_0$, on substituting in 48.4,

$$\left|\frac{V_n}{V_0}\right| = \frac{r}{\surd\left(r^2 + \frac{2Ln^2}{C}\right)}.$$

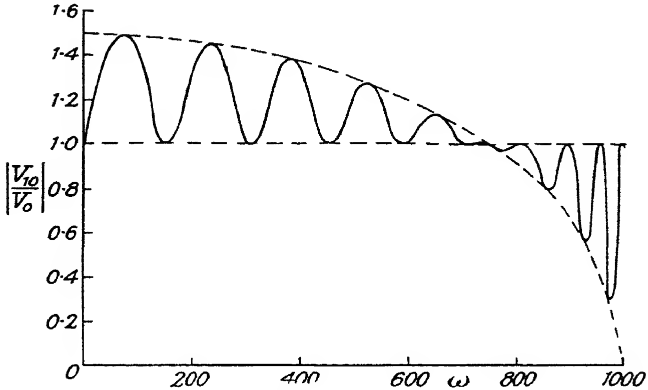


FIG. 127.

The graph of $|V_1/V_0|$ for a single section is shown in Fig. 126; for this case when $\omega = \omega_0$,

$$\left|\frac{V_1}{V_0}\right| = \frac{1500}{\surd 1500^2 + 4 \times 10^6} = \frac{3}{5}.$$

A much more striking case is shown in Fig. 127 for $n = 10$.

The sending-end impedance can be treated in the same way, its limiting lines being Z_0^2/r and r, while similar diagrams may be made for a reactive termination, though of course the arithmetical work is heavier.

§ 50. Phase Shifting Networks.

We now consider a class of networks which are rapidly coming into prominence in connection with long-distance telephony.

The general definition of filters required only that the artificial line should be composed of pure reactances. A number of artificial line types have been obtained in previous chapters in which the characteristic impedance is the geometric mean of two independent impedances. The bridge section (Chapter II., § 16), the bridged T section (Chapter II., § 18) and the special case of the generalised bridged T (Chapter III., § 28) are of this type. Suppose now we make the two impedances pure reactances reciprocal with respect to some resistance R; then the characteristic impedance will be R at all frequencies. Hence, from the argument of § 42, such an artificial line section has zero attenuation constant at all frequencies. An artificial line

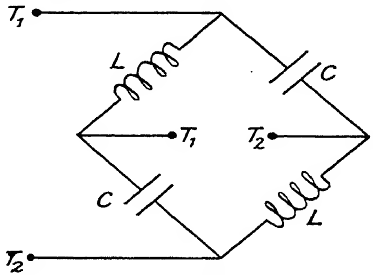


Fig. 128.

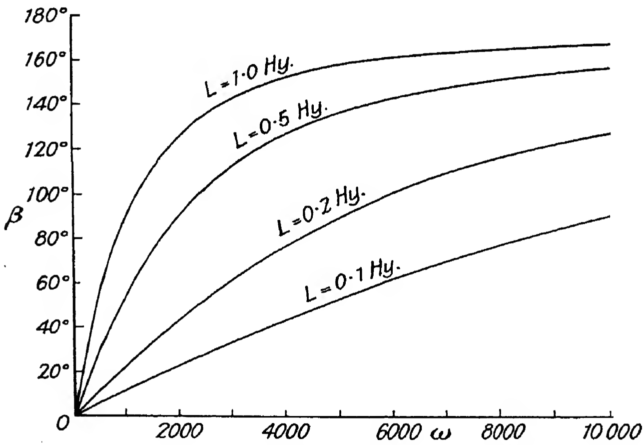


Fig. 129.

composed of a number of such sections will have the property that, if terminated by a resistance R, the current through this resistance, due to a voltage v applied at the input terminals, will

be $|v/R|$ in magnitude at all frequencies, but there will be a difference in phase.

The simplest phase shifting network that can be devised is the bridge section shown in Fig. 128.

The impedances of the inductance L and the capacity C are mutually reciprocal with respect to a resistance of value $\sqrt{L/C}$; hence the value of Z_0 is $\sqrt{L/C}$.

Further we have

$$Z_0 \tanh \theta/2 = j\omega L,$$

i.e.

$$Z_0 \tanh j\beta/2 = j\omega L,$$

or

$$j\sqrt{L/C} . \tan \beta/2 = j\omega L.$$

$$\therefore \tan \beta/2 = \omega\sqrt{LC}.$$

In Fig. 129 the value of the phase angle for a number of different sections is shown; in each case $Z_0 = 1000$ ohms, and the value of the inductance L is shown on each curve.

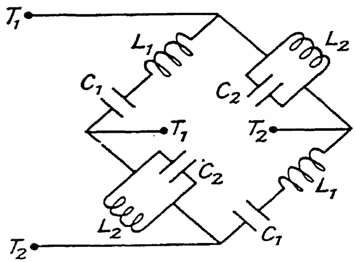


Fig. 130.

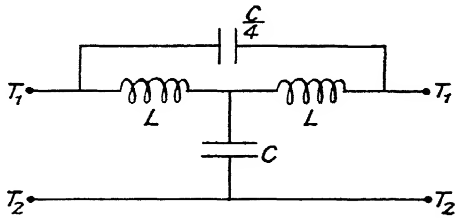


Fig. 131.

Another simple type of phase shifting bridge section is shown in Fig. 130.

If R is the characteristic impedance and the reciprocal condition holds, then

$$\left.\begin{aligned} L_2 &= C_1R^2 \\ C_2 &= \frac{L_1}{R^2} \end{aligned}\right\} \text{(Chap. III., § 25),} \qquad \text{i.e.} \qquad R^2 = \frac{L_2}{C_1} = \frac{L_1}{C_2}.$$

The phase angle is obtained from the formula

$$R \tanh j\beta/2 = j\omega L_1 + \frac{1}{j\omega C_1},$$

$$R \tan \beta/2 = \omega L_1 - \frac{1}{\omega C_1},$$

$$\tan \beta/2 = \frac{\omega L_1}{R}\left(1 - \frac{1}{\omega^2 L_1 C_1}\right).$$

Thus $\beta/2$ is $-\pi/2$ when $\omega = 0$, zero when $\omega = 1/\sqrt{L_1C_1}$, and $\pi/2$ when $\omega = \infty$.

The simplest example of phase shifter derived from the type of artificial line described in Chapter III., § 28, is shown in Fig. 131.

The characteristic impedance is

$$Z_0 = \sqrt{2L/C},$$

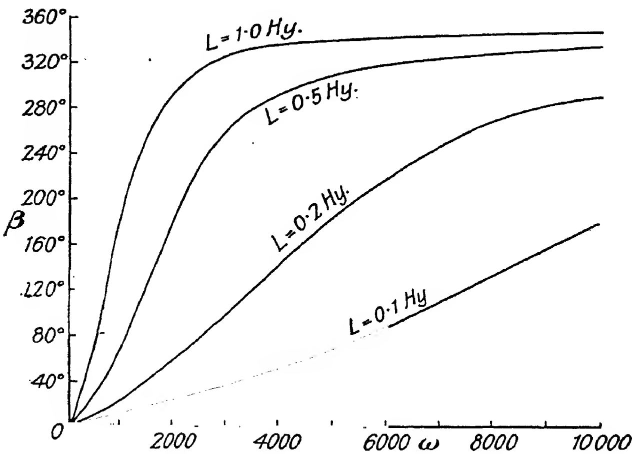


FIG. 132.

and the phase shift is given by

$$\tan \beta/2 = \frac{\omega\sqrt{2LC}}{2 - \omega^2 LC}.$$

Curves for samples of this type of phase shifter are shown in Fig. 132 ($Z_0 = 1000$ ohms).

§ 51. The Constant-Voltage-Constant-Current Properties of Filters —Boucherot's Constant Current Networks.

A number of simple networks which enable constant amplitude current to be taken from a constant potential alternating source have been described by Boucherot; they are shown in Fig. 133.

In each case L and C must be tuned to the supply frequency; if this is done for the three networks shown and if a voltage V is

applied to the terminals T_1T_2 the current through z will be equal in amplitude to $V\sqrt{C/L}$ and will be entirely independent of the value of z; the phase of the current through z will, however, generally depend on the value of z. The proof of this property of these networks is simple and need not be given.

It can be shown generally that these networks are the very simplest cases of a general property of all filters for some frequencies in their non-attenuating bands.

Let us consider the general formulæ for any filter of n sections in a non-attenuating band.

The sending-end impedance is

$$Z_0 \frac{z \cos n\beta + jZ_0 \sin n\beta}{Z_0 \cos n\beta + jz \sin n\beta} \qquad 51.1$$

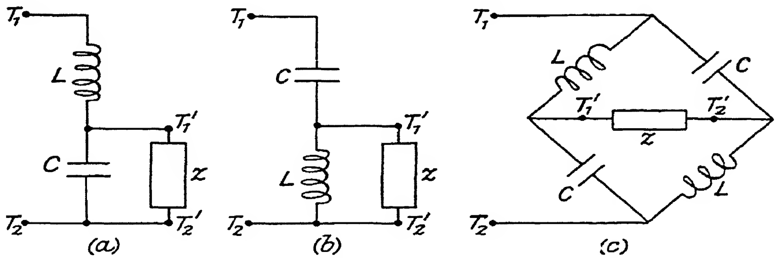


FIG. 133.

and y_n, the receiving-end impedance, is

$$y_n = jZ_0 \sin n\beta + z \cos n\beta. \qquad . \qquad . \qquad 51.2$$

For frequencies such that $\cos n\beta = 0$, $n\beta$ is of the form $m\pi \pm \pi/2$, and y_n becomes

Hence a voltage V applied to the input terminals of any filter produces, at such frequencies in the non-attenuating range, a constant output current V/Z_0, at an angle $\pm\ \pi/2$ with the input voltage, whatever the value of z.

The input current is Vz/Z_0^2; and therefore if the load is a resistance, the input power factor is unity.

Consider for example a simple low-pass T section as shown in Fig. 113, § 43 :—

$$Z_0 = \sqrt{\frac{L}{C}(2 - \omega^2 LC)}$$

and $$\cos\beta = 1 - \omega^2 LC.$$

We see that if we consider a single section its phase angle will be of the required form $m\pi \pm \pi/2$ when

$$\cos\theta = 0,$$

i.e. $$\omega^2 LC = 1,$$

$$\omega = \frac{1}{\sqrt{LC}}$$

$$= \frac{\omega_0}{\sqrt{2}}.$$

where $\omega_0 = 2\pi \times$ cut-off frequency.

If the filter consists of two simple low-pass T sections it will have the constant current property when

$$\cos 2\beta = 0,$$

i.e. when $$\cos\beta = \pm\frac{1}{\sqrt{2}}.$$

Thus the values of ω are given by

$$1 - \omega^2 LC = \pm\frac{1}{\sqrt{2}}.$$

or $$\frac{\omega}{\omega_0} = \sqrt{\frac{1}{2} \pm \frac{\sqrt{2}}{4}} = \cdot 924 \text{ or } \cdot 383.$$

Similarly, a filter of n such sections will have n constant current frequencies.

In treating the case of the simple low-pass T filter it will be seen that we have in fact dealt with the first Boucherot network (Fig. 133 (*a*)); the T section, however, containing an additional inductance in series with the load; this added inductance has the property of making the input power-factor unity for any resistance load.

In the same way the second network (Fig. 133 (*b*)) can be derived from the simple high-pass T.

It is also possible to obtain both Boucherot networks from the corresponding Π sections.

The network of Fig. 133 (*c*) is the simplest case of a phase shifting network with an angle of 90°.

For any filter the converse property holds. If at such a

frequency constant current is forced into the input terminals the output voltage is constant and independent of the load impedance z.

For if V_2 is the output voltage and I_1 the input current,

$$\frac{V_2}{I_1} = \frac{zz_n}{y_n} = \frac{Z_0 z}{Z_0 \cos n\beta + jz \sin n\beta};$$

and if $\cos n\beta = 0$,

$$V_2/I_1 = \pm jZ_0.$$

Thus suppose there are two filters connected as in Fig. 134, and that at a certain frequency both fulfil the constant current con-

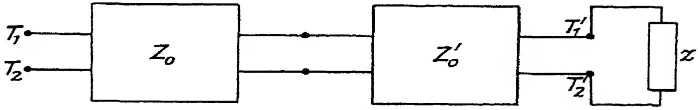


FIG. 134.

dition; let their characteristic impedances at that frequency be Z_0 and Z_0' respectively.

Then if we apply a voltage V_1 to T_1T_2 a current $V_1/\pm jZ_0$ will flow into the second filter, and therefore the output voltage across z will be given by

$$\frac{V_2}{V_1/\pm jZ_0} = \pm jZ_0'$$

i.e.

$$V_2/V_1 = \pm Z_0'/Z_0.$$

Hence at this frequency the two filters will act as a perfect transformer of ratio Z_0'/Z_0.

CHAPTER VIII.

THE HOMOGRAPHIC TRANSFORMATION AND CIRCLE DIAGRAMS.

§ 52. Terminal and other Impedance Circles. General.

We have seen in Chapter II. that the sending-end impedance of any four-terminal network is obtained from the terminal impedance by subjecting it to a homographic transformation of the form

$$z_1 = \frac{az + b}{cz + d}.$$

In this chapter we shall consider the geometrical significance of the homographic transformation and its bearing on electrical networks and artificial lines in general, showing how it accounts for the many circle diagrams that occur in circuit theory.

§ 53. Geometrical Aspect of the Homographic Transformation.

The transformation,

$$z_1 = \frac{az + b}{cz + d},$$

may be put in the form

$$z_1 - a/c = \frac{bc - ad}{c^2(z + d/c)}.$$

In general, a, b, c, d will be complex quantities, and so, in general, will z.

Let

$$\frac{bc - ad}{c^2} = \mu^2 \underline{/2\psi},$$

and let O_1 be the origin of a pair of rectangular axes, viz. a real axis and an imaginary axis. Let P_1 be the point corresponding to z. Now let the point $-d/c$ be a new origin O_2: the new axes are shown in Fig. 135 by dotted lines. Referred to the new axes P_1 is the point $z + d/c$.

We require to find

$$\frac{bc - ad}{c^2(z + d/c)}.$$

Put $z + d/c = D\angle\delta$. Then

$$\frac{bc - ad}{c^2(z + d/c)} = \frac{\mu^2\angle 2\psi}{D\angle\delta}$$
$$= \frac{\mu^2\angle 2\psi - \delta}{D}.$$

With centre O_2 draw a circle of radius μ, and let P_2 be the inverse point of P_1 with respect to this circle. Draw a line O_2Q making an angle ψ with the positive direction of the real axis, and reflect P_2 in the line O_2Q to give the point P_3.

Consider what the point P_3 represents in the complex plane with O_2 as origin.

The length of O_2P_3 is μ^2/O_2P_1, that is, μ^2/D, and the angle O_2P_3 makes with the real axis is

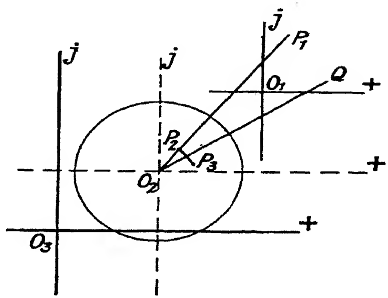


FIG. 135.

$$\psi - (\delta - \psi) = 2\psi - \delta.$$

Thus P_3 is the required point

$$\frac{bc - ad}{c^2}(z + d/c),$$

with O_2 as origin, and is therefore the point

$$z_1 - a/c.$$

Now change the origin to O_3, a point $-a/c$ referred to the axes through O_2, and draw a third pair of axes, shown in Fig. 135 by the heavy lines.

P_3 is now the required point z_1, O_3 being taken as origin.

Thus given any network we can, for any one frequency, by a simple geometrical construction derive the sending-end impedance.

The procedure is:—

(1) Draw a pair of rectangular axes with origin O_1.

(2) Draw a second pair of axes parallel to the first pair having

an origin O_2, where O_2 is the point $-d/c$ referred to the original axes through O_1.

(3) Draw a circle with O_2 as centre and radius

$$\mu = \sqrt{\left|\frac{bc - ad}{c^2}\right|}.$$

(4) Draw a line O_2Q, making an angle with the positive axis of reals equal to ψ, i.e. half the angle of

$$\frac{bc - ad}{c^2}.$$

(5) Draw a third pair of axes parallel to the previous ones having an origin O_3, where O_3 is the point $-a/c$ referred to the axes through O_2.

Having completed this construction, to find the sending-end impedance of the network terminated by an impedance z proceed as follows:—

(1) Plot the point P_1, representing z, with O_1 as origin.

(2) Find P_2, the inverse point of P_1 with respect to the circle, with centre O_2.

(3) Reflect the point P_2 in the line O_2Q, obtaining the point P_3.

(4) The point P_3 referred to the axes through O_3 is the required sending-end impedance.

The process may appear complicated, but when once mastered can be carried out on a drawing board very quickly, and enables the general behaviour of any network to be studied rapidly and with a great saving of arithmetical labour.

§ 54. The Homographic Transformation applied to a General Network.

For a network, such as is discussed in Chapter II., having constants Z, θ and ϕ, the impedance measured at T_1T_2 is

$$z_1 = Z\frac{z \cosh(\theta + \phi) + Z \sinh \theta}{z \sinh \theta + Z \cosh(\theta - \phi)} \quad . \quad . \quad 54.1$$

Comparing it with the standard form of transformation of the previous section we see that

$$\left.\begin{aligned} a &= Z \cosh(\theta + \phi) \\ b &= Z^2 \sinh \theta \\ c &= \sinh \theta \\ d &= Z \cosh(\theta - \phi) \end{aligned}\right\} \quad . \quad . \quad . \quad 54.2$$

In the geometrical construction the first step is to establish a new origin at the point $-d/c$.

From 54.2

$$-\frac{d}{c} = -\frac{Z\cosh(\theta - \phi)}{\sinh\theta} \quad . \quad . \quad . \quad 54.3$$

and a comparison with 13.19, Chap. II., shows that this is $-Z_f'$.

Now Z_f', the impedance of the network measured at $T_1'T_2'$ with T_1T_2 open-circuited, is the impedance of a passive network and therefore its real part must be zero or positive. Hence $-d/c$ has its real part zero or negative and therefore O_2 cannot lie on the right-hand side of O_1 in Fig. 135.

Consider next the other change of origin—

$$-\frac{a}{c} = -\frac{Z\cosh(\theta + \phi)}{\sinh\theta} \quad . \quad . \quad . \quad 54.4$$

Comparing with formula 13.19, Chap. II., we see that $\dfrac{Z\cosh(\theta + \phi)}{\sinh\theta}$ is Z_f, the impedance of the network measured at T_1T_2 with $T_1'T_2'$ free. From this we deduce that $-a/c$ has always its real part negative or zero, and thus in Fig. 135 O_3 cannot lie on the right-hand side of O_2.

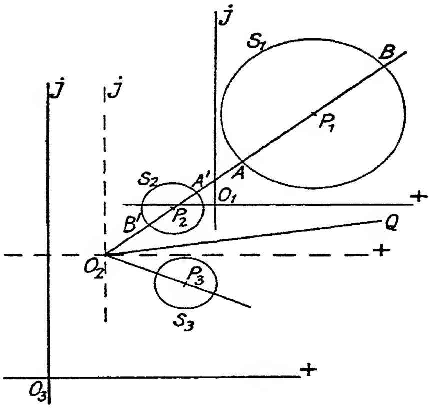


FIG. 136.

We may note in passing that, referred to O_1 as origin, O_3 is the point

$$-(Z_f + Z_f'),$$

i.e. the point

$$-2Z\coth\theta\cosh\phi.$$

If the network is symmetrical, so that $\phi = 0$, then

$$\frac{a}{c} = \frac{d}{c} = Z_0\coth\theta,$$

and therefore O_1, O_2, and O_3 are collinear, and O_2 is the mid-point of O_1O_3.

We now require to examine the value of

$$\frac{bc - ad}{c^2}.$$

It is equal to

$$\frac{Z^2\{\sinh^2\theta - \cosh(\theta+\phi)\cosh(\theta-\phi)\}}{\sinh^2\theta},$$

which readily reduces to

$$-Z^2\frac{\cosh^2\phi}{\sinh^2\theta}, \qquad . \quad . \quad . \quad . \qquad 54.5$$

and if the network is symmetrical it simplifies to

$$-\frac{Z_0{}^2}{\sinh^2\theta} \qquad . \quad . \quad . \quad . \qquad 54.6$$

§ 55. The Circle Property of the Homographic Transformation.

We have seen that the homographic transformation involves two changes of origin, an inversion and a reflection. Suppose that z moves round the circumference of a circle, then it is clear that z_1 must move round the circumference of a circle since the inverse of a circle is a circle and the changes in origin and reflection cannot affect the shape of the curve; of course in this we must regard a straight line as a circle of infinite radius.

We shall now proceed to solve the problem :—

Given that z describes a given circle, determine the circle that z_1 describes in the z_1-plane.

As described in § 53, let O_1, O_2, O_3 be the origins of the systems of axes.

Let z describe a circle of radius r about a point P_1, which, referred to the axes through O_1, is the point $x + jy$; let this circle be denoted by S_1.

Let

$$\left.\begin{aligned} d/c &= x' + jy' \\ \text{and} \qquad a/c &= x'' + jy'' \end{aligned}\right\}. \qquad . \quad . \quad . \qquad 55.1$$

Let O_2P_1 cut the circle S_1 in the points A and B and let A′ and B′ be the inverse points of A and B with respect to the circle, with O_2 as centre and radius μ. Let S_2 denote a circle with A′B′ as diameter. Then S_2 will be the circle obtained by inverting S_1 with respect to the circle of centre O_2 and radius μ.

To complete the construction it is necessary to reflect the circle S_2 in O_2Q and obtain the final circle S_3; this does not alter the linear dimensions, so that A′B′ is equal to twice the radius of S_3.

Referred to O_2 as origin P is the point

$$(x + x') + j(y + y')$$

and if

$$D^2 = (x + x')^2 + (y + y')^2 \qquad . \quad . \quad . \qquad 55.2$$

the length of O_2P_1 is equal to D and the lengths of O_2A and O_2B are respectively $D - r$ and $D + r$.

Hence the lengths of O_2A' and O_2B' are respectively

$$\frac{\mu^2}{D - r} \quad \text{and} \quad \frac{\mu^2}{D + r},$$

so that the diameter of S_2 and therefore of S_3 is

$$\begin{aligned} A'B' &= \mu^2\left(\frac{1}{D - r} - \frac{1}{D + r}\right) \\ &= \mu^2\frac{2r}{D^2 - r^2} \\ &= \left|\frac{bc - ad}{c^2}\right| \cdot \frac{2r}{D^2 - r^2} \end{aligned} \qquad 55.3$$

A simple corollary is that the diameter of S_2 and therefore of S_3 depends only on the radius of S_1 and the distance of its centre from O_2.

It is now necessary to determine the position of the centre of S_3 referred to O_3 as origin.

The distance of the centre P_2 of S_2 from O_2 is

$$\tfrac{1}{2}(O_2A' + O_2B'),$$

which is equal to

$$\tfrac{1}{2}\mu^2\left(\frac{1}{D - r} + \frac{1}{D + r}\right) = \frac{\mu^2 D}{D^2 - r^2}.$$

Let
$$\frac{y + y'}{x + x'} = \tan\delta.$$

Then P_2 is the point

$$\frac{\mu^2 D}{D^2 - r^2}\underline{/\delta}$$

referred to O_2 as origin.

The centre of S_3 is the reflection of P_2 in OQ and is therefore the point

$$\frac{\mu^2 D}{D^2 - r^2}\underline{/2\psi - \delta}$$

referred to O_2 as origin.

On making the final change of origin to O_3, P_3 the centre of S_3 is the point

$$\frac{\mu^2 D}{D^2 - r^2}\underline{/2\psi - \delta} + x'' + jy'', \qquad 55.4$$

or

$$\left\{\frac{\mu^2 D}{D^2 - r^2}\cos(2\psi - \delta) + x''\right\} + j\left\{\frac{\mu^2 D}{D^2 - r^2}\sin(2\psi - \delta) + y''\right\},$$

and therefore 55.3 and 55.4 give the complete solution of the problem undertaken.

Two important properties of circles and their inverses which will be required later on may be noted here:—

A circle S_1 is inverted with respect to a circle whose centre is X giving a circle S_2. Then

(1) if X lies outside the circle S_1 all points within the circle S_1 will invert into points within the circle S_2, and

(2) if X lies inside the circle S_1, all points within the circle S_1 will invert into points outside the circle S_2.

§ 56. The Terminal Impedance Circle of an Electrical Network.

The terminal impedance z is a passive network, i.e. the real part of its impedance is either zero or is positive; thus z must lie in the half of the plane to the right of the imaginary axis through O_1.

The imaginary axis through O_1 can be regarded as the circumference of the circle of infinite radius whose centre is at the point $+\infty$ on the real axis. Thus any value of z whose real part is not negative must lie within this circle.

The infinite circle we take as the circle S_1; on inverting we obtain the circle S_2. Since O_2 lies outside S_1 (for we have already proved that O_2 is to the left of O_1), all points within S_1 will invert into points inside S_2 and finally, on reflection, become points inside S_3. Hence the sending-end impedance, whatever the value of the terminating impedance, must lie within the circle S_3 so obtained, and if z is a pure reactance it must lie on the circumference of S_3.

The circle S_3 is called the *Terminal Impedance Circle*. It is obtained simply by inverting the imaginary axis through O_1 with respect to the circle of radius μ and centre O_2 and then reflecting the circle so obtained in O_2Q.

From the formulæ of the last section we can calculate the diameter and position of the centre.

To obtain the diameter put

$$x = r = h \to \infty, \qquad . \quad . \quad . \quad 56.1$$

$$y = 0,$$

$$D^2 = (h + x')^2 + y'^2.$$

From 55.3 the diameter is given by

$$\lim_{h \to \infty} \mu^2 \frac{2h}{(h + x')^2 + y'^2 - h^2} = \frac{\mu^2}{x'} . \qquad 56.2$$

To obtain the centre

$$\delta = 0 \text{ when } h \text{ is infinite.}$$

$$\therefore \frac{D}{D^2 - r^2} = \lim_{h \to \infty} \frac{\sqrt{(h + x')^2 + y'^2}}{(h + x')^2 + y'^2 - h^2}$$

$$= \frac{1}{2x'} .$$

Hence the centre is the point

$$\left.\begin{aligned} &\frac{\mu^2}{2x'} \underline{/2\psi} + x'' + jy'' \\ &= \frac{1}{2x'}\left(\frac{bc - ad}{c^2}\right) + x'' + jy'' \end{aligned}\right\} \qquad 56.3$$

From this equation we deduce that O_2Q is the bisector of the angle that the line joining the centre to O_2 makes with the positive real axis.

The graphical construction of the terminal impedance circle is very simply carried out on a drawing board.

§ 57. Terminal Impedance Circles. Simple Examples.

A simple case of frequent occurrence in the literature of coupled circuits is where a resistance R is shunted by an impedance z (Fig. 137).

From first principles

$$z_1 = \frac{Rz}{R + z} .$$

Here
$$a = R,$$
$$b = 0,$$
$$c = 1,$$
$$d = R.$$

So that
$$d/c = R,$$
$$x' = R, \quad y' = 0,$$
$$a/c = R,$$
$$x'' = R, \quad y'' = 0.$$
$$\frac{bc - ad}{c^2} = - R^2,$$

hence
$$\psi = \pi/2,$$
$$\mu = R.$$

From 56.2 the diameter of the impedance circle is

$$\mu^2/x' = R^2/R = R$$

and the centre is the point

$$\frac{\mu^2 \angle 2\psi}{2x'} + x'' + jy''$$
$$= -R^2/2R + R$$
$$= R/2.$$

Thus the terminal impedance circle is as shown in Fig. 138, so that, whatever the value of the impedance z, the impedance measured at T_1T_2 must lie either on or within this circle; it must lie on the circle if z is a pure reactance; for example, we see that the reactance of the network must lie within the limits $\pm jR/2$ ohms.

Another simple case is shown in Fig. 139.

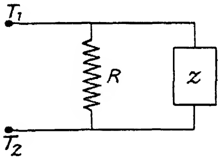


FIG. 137.

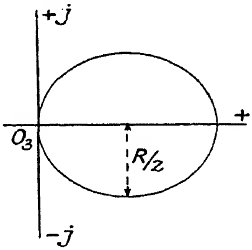


FIG. 138.

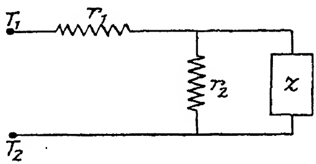


FIG. 139.

The sending-end impedance is

$$z_1 = r_1 + \frac{r_2 z}{r_2 + z}$$
$$= \frac{z(r_1 + r_2) + r_1 r_2}{z + r_2}.$$

Here

$$a = r_1 + r_2,$$
$$b = r_1 r_2,$$
$$c = 1,$$
$$d = r_2,$$
$$d/c = r_2,$$
$$x' = r_2, \quad y' = 0,$$
$$a/c = r_1 + r_2,$$
$$x'' = r_1 + r_2, \quad y'' = 0,$$
$$\frac{bc - ad}{c^2} = -r_2^2,$$

hence the radius of the impedance circle is $r_2/2$, and the centre is the point $r_1 + r_2/2$ on the positive real axis.

§ 58. The Terminal Impedance Circle of an Artificial Line.

The case of an artificial line of n sections can be deduced from the equations of § 54 by replacing θ by $n\theta$ and putting $\phi = 0$.

Hence

$$a/c = d/c = Z_0 \coth n\theta,$$
$$\frac{bc - ad}{c^2} = -\left(\frac{Z_0}{\sinh n\theta}\right)^2.$$

Let

$$Z_0 = X + jY.$$

Then

$$\begin{aligned} x' + jy' &= x'' + jy'' \\ &= Z_0 \coth n\theta \\ &= (X + jY) \coth n(\alpha + j\beta) \\ &= \frac{X \sinh 2n\alpha + Y \sin 2n\beta}{2(\cosh^2 n\alpha - \cos^2 n\beta)} \\ &\quad + j\frac{Y \sinh 2n\alpha - X \sin 2n\beta}{2(\cosh^2 n\alpha - \cos^2 n\beta)}, \end{aligned} \qquad 58.1$$

and

$$\begin{aligned} \mu^2 &= \left|\frac{(X + jY)^2}{\sinh^2 n\theta}\right| \\ &= \frac{X^2 + Y^2}{\cosh^2 n\alpha - \cos^2 n\beta}. \end{aligned} \qquad 58.2$$

Hence if the radius of the terminal impedance circle, which by 56.2 is $\mu^2/2x'$, is ρ, it is given by

$$\rho = \frac{X^2 + Y^2}{X \sinh 2n\alpha + Y \sin 2n\beta}. \qquad 58.3$$

Suppose now that a curve is drawn in which the radius of the impedance circle is plotted against n for a given fixed frequency.

In the case of the artificial line we are interested only in integral values of n, but the same formulæ hold for a uniform line if n is replaced by l, the length of line, and θ by P, the propagation constant of the line.

The numerator of ρ is constant; in the denominater $X \sinh 2n\alpha$ will steadily increase as n is increased, but the second term will oscillate from $-Y$ to $+Y$.

Hence as n increases ρ will in general diminish, but will also undulate between the two curves

$$\rho = \frac{X^2 + Y^2}{X \sinh 2n\alpha \pm Y};$$

these two limiting curves will approach one another as n increases.

The general effect is shown in Fig. 140, which, however, is not drawn to any scale; it is closely related to the well-known Ferranti Effect in cables.

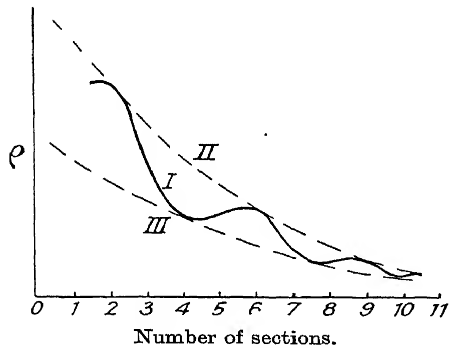
The co-ordinates of the centre are given by

$$\frac{1}{2} \cdot \frac{-\dfrac{Z_0^2}{\sinh^2 n(\alpha + j\beta)}}{\text{Real part of } Z_0 \coth n\theta} + Z_0 \coth n\theta.$$

Replacing Z_0 by $X + jY$ this reduces readily to

$$\frac{X^2 \cosh 2n\alpha + Y^2 \cos 2n\beta + jXY(\cosh 2n\alpha - \cos 2n\beta)}{X \sinh 2n\alpha + Y \sin 2n\beta}.$$

Consider the effect of increasing n, the number of sections; $\cosh 2n\alpha$ and $\sinh 2n\alpha$ will increase with n, but $\cos 2n\beta$ and $\sin 2n\beta$ will vary periodically between 1 and -1. Hence, if we plot a curve showing the real co-ordinate of the centre with n as abscissa we shall obtain a curve undulating about $X \coth 2n\alpha$ in a very similar way to the undulation occurring in the case of the radius shown in Fig. 140. Similarly, the imaginary co-ordinate undulates about $Y \coth 2n\alpha$. Ultimately as n becomes very large the centre approaches $Z_0 \coth 2n\alpha$ and when n is infinite the centre reaches the point Z_0. If three axes at right angles, one of n and the other two of resistance and reactance, were taken it would be



Curve I is $\rho = \dfrac{X^2 + Y^2}{X \sinh 2n\alpha + Y \sin 2n\beta}$,

Curve II is $\rho = \dfrac{X^2 + Y^2}{X \sinh 2n\alpha - Y}$,

Curve III is $\rho = \dfrac{X^2 + Y^2}{X \sinh 2n\alpha + Y}$.

FIG. 140.

possible to construct a curve on which the centre of the terminal impedance circle lies; it would be of the form of a bent and diminishing corkscrew.

§ 59. Constant Resistance and Constant Reactance Circles.

It has already been seen that the first step in obtaining the terminal impedance circle is to invert the imaginary axis with respect to a circle, centre O_2 and radius μ; we then obtain a circle passing through O_2, the centre of this circle lying on the line through O_2 parallel to the real axis; this circle we have called S_2.

Now suppose that the terminal impedance is specified as having a fixed resistance r_1 and a reactance which may vary from $-\infty$ to $+\infty$. Then z will lie on a straight line parallel to the imaginary

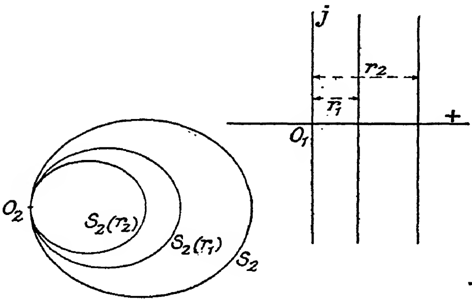


FIG. 141.

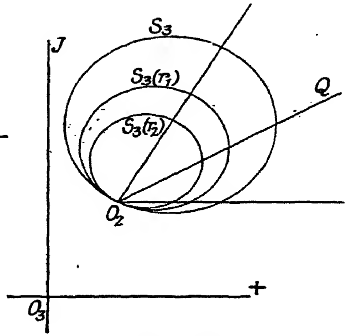


FIG. 142.

axis at a distance r_1 to the right of O_1. If we invert this line with respect to the circle, centre O_2 and radius μ, we obtain a circle which touches S_2 at O_2 and lies within S_2; call this circle $S_2(r_1)$.

Similarly, by inverting each line of constant resistance we can construct a whole series of circles $S_2(r_2)$, $S_2(r_3)$, etc., all of which will touch S_2 at O_2; this procedure gives us Fig. 141.

If we now reflect S_2, $S_2(r_1)$, $S_2(r_2)$, etc., in O_2Q and change the origin to O_3 we obtain a set of circles S_3, $S_3(r_1)$, $S_3(r_2)$, etc., as shown in Fig. 142 which has been drawn apart from Fig. 141 to avoid confusion: S_3 is, of course, the terminal impedance circle.

Consider the significance of these circles; if z lies on the imaginary axis through O_1, i.e. if z is a pure reactance, then z must lie on the circle S_3. If z lies on the line parallel to the imaginary axis

at a distance r_1 to the right of it, i.e. if z has a constant resistance component r_1 and a variable reactance component, then z must lie on the circle $S_3(r_1)$.

Hence we can map out the interior of the terminal impedance circle by a set of circles all touching it at the point O_2, each circle corresponding to a definite resistance r. If r is the real part of z, z must lie somewhere on the circumference of the circle $S_3(r)$.

In an exactly similar way we can construct a series of constant reactance circles.

Suppose, firstly, that z is non-reactive, then z must lie on the real axis through O_1, and on inverting the real axis with respect to the circle, centre O_2 and radius μ, we obtain the circle S_2'.

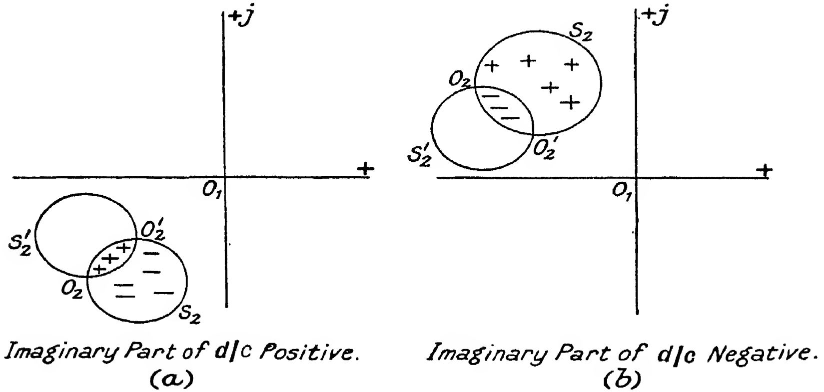


FIG. 143.

There are two cases according as the imaginary part of d/c is positive or negative, i.e. according as O_2 is below or above O_1. Take first the case where the imaginary part of d/c is positive; O_2 is below O_1. Then S_2', the inverse of the real axis, cuts the circle S_2, the inverse of the imaginary axis, orthogonally at O_2 and O_2', where O_2' is the other point of intersection.

Now all values of z, which have positive reactances, must lie above the real axis through O_1. They must, therefore, invert into points within the circle S_2'. Similarly, all values of z where the reactances are negative must invert into points outside the circle S_2'. Fig. 143 shows how S_2 is divided by S_2' depending on whether the reactance of z is + or −.

After reflection and change of origin we obtain the circle S_3', which cuts the terminal impedance circle orthogonally at O_2 and O_2''.

If z has a positive reactance, z_1 must lie inside both the circles S_3 and S_3'; if z has a negative reactance z_1 must lie in that part of the circle S_3 which is outside the circle S_3'.

For the second case, where d/c has a negative imaginary part we see that if z has a positive reactance, z_1 must lie inside the circle S_3 and outside the circle S_3'.

The circle S_3' might be called the *Terminal Zero Reactance Circle ;* its radius and centre can be calculated in exactly the same way as in the case of the terminal impedance circle; an extremely interesting case of a Terminal Zero Reactance Circle is given in Ex. 7 at the end of the chapter.

Just as we constructed constant resistance circles we can construct constant reactance circles; after inverting and reflecting, we get a set of circles $S_3'(jr_1')$, $S_3'(jr_2')$, etc., all of which touch the terminal zero reactance circle at O_2; every constant resistance circle cuts every constant reactance circle orthogonally.

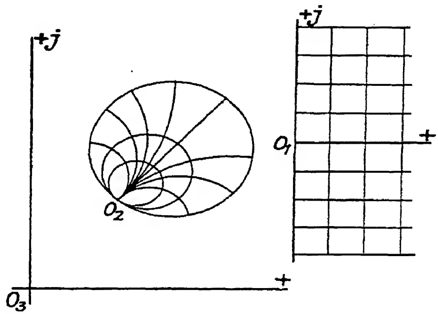


FIG. 144.

Thus by means of the two systems of constant resistance circles $S_3(r)$ and constant reactance circles $S_3'(jr')$, we can map out the area of the terminal impedance circle as shown in Fig. 144, each element of area of the circle S_3 in the z_1-plane, bounded by four orthogonal circular arcs, corresponding to a rectangular element in the right half of the z-plane. This is, of course, a simple case of Conformal Representation.

When this system of circles has been constructed z_1, corresponding to each value of z, can be determined immediately; thus if $z = x + jy$ then z_1 is given by the point in the terminal impedance circle where the constant resistance circle corresponding to x cuts the constant reactance circle corresponding to y.

Of the constant reactance circles one will be a straight line and

a diameter of S_3—the corresponding constant value of reactance will be equal and opposite to that of d/c.

§ 60. Constant Terminal Angle Circles.

Suppose z moves along a straight line through O_1, i.e. let the terminating impedance be variable but have a constant angle or power factor. A little consideration will show that the corresponding locus for . is the arc of the circle which passes through O_2 and O_2'', cutting the terminal impedance circle at an angle equal to the angle that the locus of z through O_1 makes with the imaginary axis; Fig. 145 shows the arcs for $+\pi/4$ and $-\pi/4$ in the case where the imaginary part of d/c is positive; arc I corresponds to $+\pi/4$, arc II to $-\pi/4$.

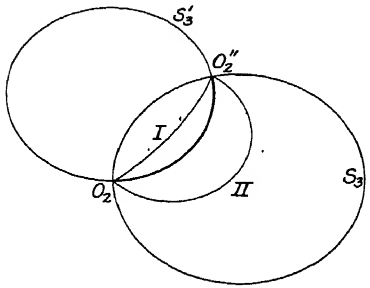


Fig. 145.

In this way the area within the terminal impedance circle can be divided into lunes by a series of constant terminal angle circles.

The chord O_2O_2'' represents in the z_1-plane all impedances in the z-plane having a constant angle equal to that of d/c: the zero reactance circle is the circle of zero terminal angle.

§ 61. Frequency Impedance Circles.

In general, for an electrical network, the homographic transformation which is applied to z in order to obtain the sending-end impedance z_1 depends on frequency, and thus the values of the radius and the co-ordinates of the centre of the terminal impedance circle depend on frequency. But if the network consists entirely of resistances the radius and centre are independent of frequency: moreover, μ, ψ and the points O_2 and O_3 are also independent of frequency.

If such a network is terminated by any fixed impedance z and the frequency varied and we then carry out the graphical process of § 53 to determine z_1 we find that z_1 must lie within the terminal impedance circle of the network, so that in this case the terminal impedance circle can be called the *Frequency Impedance Circle*. If z is a pure reactance z_1 lies on the circumference of the frequency impedance circle at all frequencies.

§ 62. Terminal Impedance Admissible Areas.

In the last section it has been shown that if a network consists entirely of resistances, a single circle can be drawn such that whatever the frequency or the value of z, z_1 lies within or on the circle.

In the more general case, however, where the circle varies with frequency, its envelope will trace out an area, which may be called the *Terminal Impedance Admissible Area*, such that whatever the frequency or the value of z, z_1 must lie within the area.

The general determination of the envelope is impossible, but two simple cases will be considered.

First consider a single T section in which A is a resistance R and B a capacity C. The centre of the circle is the point

$$r + \frac{1}{2r\omega^2 C^2} - \frac{j}{\omega C}$$

and its radius is

$$\frac{1}{2r\omega^2 C^2}.$$

Let x and y be the co-ordinates of the centre, the imaginary axis being the axis of y, and the real axis the axis of x; then

$$x = r + \frac{1}{2r\omega^2 C^2}$$

$$y = -\frac{1}{\omega C}.$$

Eliminating ω we have

$$x = r + \frac{1}{2r}y^2,$$

which is a parabola whose vertex is at the point $(r, 0)$ and whose semi-latus rectum is r; the centre of the circle lies on the lower half of this parabola. This is shown in Fig. 146 in which A is the vertex and S the focus of the parabola and

$$O_3A = r,$$
$$AS = r/2.$$

At a definite frequency ω, the centre of the terminal impedance circle will be the point H on the parabola. The directrix is a line parallel to the y-axis bisecting O_3A. Draw HN, the perpendicular from H to the directrix; construct a semicircle with S as centre and AS as radius to lie in the lower half of the plane,

and let it cut SH at M and the real axis again at A′. Draw the tangent at the vertex.

The radius of the terminal impedance circle is $1/2r\omega^2C^2$, and therefore since the x co-ordinate of H is $r + 1/2r\omega^2C^2$ and $O_3A = r$, the terminal impedance circle will touch the tangent at the vertex for all values of ω.

Next consider the length of MH. By the definition of a parabola

$$\mathrm{SH} = \mathrm{HN} = \frac{r}{2} + \frac{1}{2r\omega^2C^2}.$$

Therefore, since $\mathrm{SM} = r/2$,

$$\mathrm{HM} = \frac{1}{2r\omega^2C^2},$$

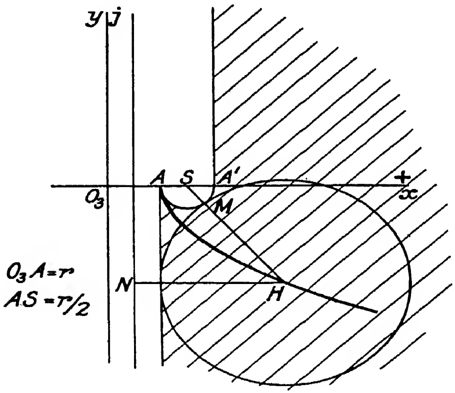


FIG. 146.

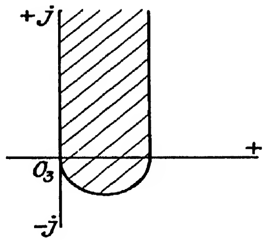


FIG. 147.

which is the radius of the terminal impedance circle; thus this circle must also touch the semicircle AMA′ for all values of ω. Thus the shaded area in Fig. 146 gives the terminal impedance admissible area.

As a second example, consider a single T section in which A is an inductance L, and B a resistance r.

The radius of the circle is $r/2$ and its centre is

$$r/2 + j\omega L.$$

Hence the admissible area is, as shown in Fig. 147, an infinite strip, of width r, in the first quadrant together with a semicircle in the fourth quadrant.

By taking three perpendicular axes, one of frequency and the other two of resistance and reactance respectively, we can construct solid models of what may be termed Admissible Spaces; thus, for the example last considered the model would be a cylinder of elliptic cross-section.

§ 63. Filters and Purely Reactive Networks.

If the network is purely reactive d/c is imaginary and O_2 lies on the imaginary axis through O_1. When we proceed to invert the imaginary axis with respect to the point $-d/c$, we are inverting a line with respect to a point on it, in which case the inverse is the line itself: thus the terminal impedance circle becomes of infinite radius. The constant resistance circles, however, are of finite radius for all finite values of resistance.

EXAMPLES.

1. Prove that, as β varies from $-\pi/2$ to $\pi/2$, while α remains constant, $\tanh(\alpha + j\beta)$ traces out a circle in the complex plane, the centre being at the point $\coth 2\alpha$ and the radius being $1/\sinh 2\alpha$.

Note that

$$\tanh(\alpha + j\beta) = \frac{j \tan\beta + \tanh\alpha}{j \tan\beta \tanh\alpha + 1},$$

i.e. $\tanh(\alpha + j\beta)$ is obtained by applying a homographic transformation to $j\tan\beta$. Similarly $\coth(\alpha + j\beta)$ can be obtained by applying a homographic transformation to $j\tan\beta$. These constructions can be carried out graphically and should be compared with the Kennelly Chart Atlas.

2. The square of the characteristic impedance of a uniform transmission line is given by

$$Z_0^2 = \frac{R + j\omega L}{S + j\omega C},$$

and is therefore obtained by applying a homographic transformation to ω as ω moves along the real axis from 0 to $+\infty$.

Show that Z_0^2 lies on a circle whose centre is the point $\frac{1}{2}(R/S + L/C)$ on the real axis, and whose radius is $\pm\frac{1}{2}(R/S - L/C)$. (This gives a simple graphical method of obtaining Z_0.)

3. Show, in the case of the first simple T network considered in § 62, that unless $\omega < 1/2rC$ the network cannot have a finite positive reactance, whatever the value of the terminating impedance.

4. The sending-end impedance of a filter for a transmitting band terminated by an impedance z is

$$z_n = Z_0 \frac{z \cos n\theta + jZ_0 \sin n\theta}{Z_0 \cos n\theta + jz \sin n\theta}$$
$$= \frac{Z_0 z + jZ_0^2 \tan n\theta}{Z_0 + jz \tan n\theta},$$

and thus is obtained by applying a homographic transformation to $\tan n\theta$. As n varies, $\tan n\theta$ moves along the real axis in the complex plane. Hence show that the points z_1, z_2, z_3, etc., lie on a circle.

Show also that for a frequency in an attenuating band the points z_1, z_2, z_3, etc., also lie on a circle.

5. If a network has three pairs of terminals, and if two external impedances are connected to two pairs of terminals and are varied together so that the impedance measured at the third pair of terminals remains constant—show that the two impedances are related homographically.

6. Show for an artificial line of which the sections consist entirely of resisttances that z_1, z_2, z_3, etc., the sending-end impedances of 1, 2, 3, etc., sections terminated by a fixed impedance, lie on the circumference of a circle. Compare with Ex. 7 of Chap. II.

7. If, in Fig. 49 of Ex. 4, Chap. II, T_1' T_2' are taken as input terminals and E and D are pure reactances such that $E = -2D$, then the Terminal Zero Reactance Circle is a circle with the origin O_3 as centre.

Hence, if a voltage is applied to T_1' T_2' and a variable resistance is connected across T_1T_2, the input current will remain constant in amplitude and vary only in phase as the resistance is varied from 0 to ∞.

(H. M. Turner, "Proc. I.R.E.," Nov. 1928.)

Show also that every filter has this property for frequencies in its transmitting bands such that

$$n\theta = m\pi \pm \pi/4$$

where m is any integer.

CHAPTER IX.

THE GENERAL THEORY OF THE MULTISTAGE THERMIONIC VALVE AMPLIFIER.

§ 64. The Single Valve.

Up to the present passive networks alone have been dealt with, but the problem of the multistage amplifier can be treated by a slight extension of the theory of repeated networks given in Chapter II.

In dealing with a single valve the equivalent circuit shown in Fig. 148(*a*) is in general use. There is, however, an alternative equivalent network shown in Fig. 148(*b*) in which the sine wave surrounded by two circles represents a constant current generator giving a constant current me/ρ, where m and ρ are the amplification

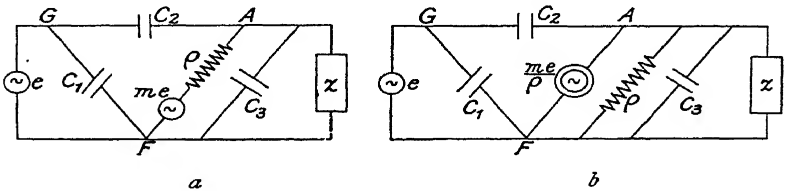


FIG. 148.

constant and internal resistance of the valve respectively, while e is the voltage applied between grid and filament. This alternative network has the very great advantage over that of Fig. 148(*a*) that it throws ρ, C_3 and the external anode load impedance all in parallel as a shunt load on the constant current generator. This alternative form is used by Butterworth ("Experimental Wireless," June, 1929).

The equivalence of the two networks depends on the following easily proved relation between constant current and constant voltage generators:—

A constant voltage generator of voltage v having an internal series resistance r is identical in action to a constant current generator giving a constant current v/r and having an internal shunt resistance r.

Let the impedance of the grid anode capacity be represented by a and let the impedance of ρ and C_3 in parallel be b, so that

$$\frac{1}{a} = j\omega C_2$$

$$\frac{1}{b} = \frac{1}{\rho} + j\omega C_3.$$

Then omitting the grid filament capacity C_1, the network to be dealt with takes the simple form shown in Fig. 149, where k is written for m/ρ and is thus what is usually termed the Mutual Conductance of the valve.

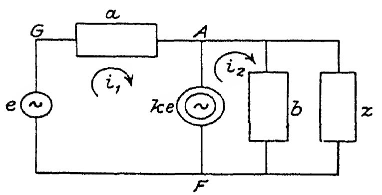


Fig. 149.

Assign circulating currents i_1 and i_2 to the first two meshes of this network; the applied voltage e must be equal to the sum of the voltage drops across a and combined impedance of b and z. Hence

$$e = ai_1 + \frac{i_2 bz}{b + z} \quad . \quad . \quad . \quad 64.1$$

The algebraic sum of the three currents flowing into the junction point A must be zero, i.e.

$$i_1 - ke - i_2 = 0$$

$$i_1 - i_2 = ke \quad . \quad . \quad . \quad 64.2$$

These two equations give i_1 and i_2

$$i_1 = e \cdot \frac{z(1 + bk) + b}{z(a + b) + ab}.$$

Thus the input impedance z_1 is given by

$$z_1 = \frac{e}{i_1} = \frac{z(a + b) + ab}{z(1 + bk) + b} \quad . \quad . \quad 64.3$$

This is of course only that part of the input impedance due to the presence of grid anode capacity—it will be called the *Additional Input Impedance*—to obtain the actual input impedance the grid filament capacity C_1 must be added in parallel.

Consider next the current through z; from 64.1 and 64.2

$$i_2 = \frac{e(1 - ak)(b + z)}{z(a + b) + ab} \quad . \quad . \quad . \quad . \quad 64.4$$

Of this current a fraction $\frac{b}{b + z}$ will flow through z so that the current through z will be

$$\frac{e(1 - ak)b}{z(a + b) + ab},$$

and the receiving-end impedance y_1 will be given by

$$y_1 = \frac{z(a + b) + ab}{(1 - ak)b} \quad . \quad . \quad . \quad . \quad 64.5$$

The voltage amplification, i.e. the ratio of the voltage across z to the input voltage e will be

$$\frac{z(1 - ak)b}{z(a + b) + ab} \quad . \quad . \quad . \quad . \quad 64.6$$

§ 65. The Direct Coupled Multistage Amplifier.

A direct coupled amplifier will be defined as one, such as a simple tuned anode or resistance amplifier, with a coupling con-

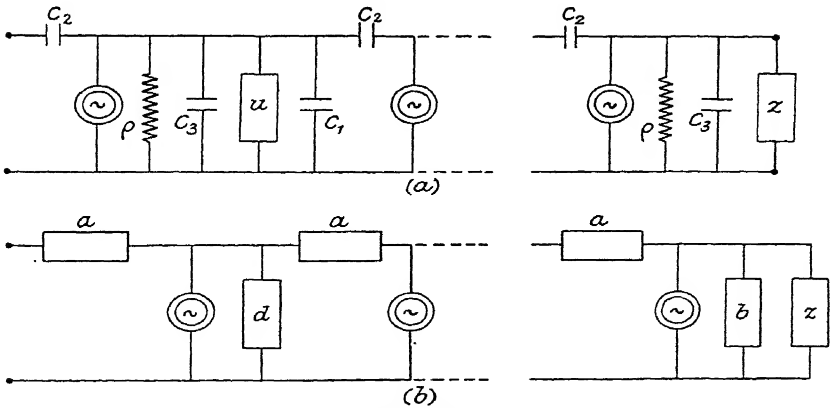


FIG. 150.

denser so large that its impedance may be neglected; the impedance in the anode of one valve is then electrically in parallel with the grid filament impedance of the following valve. The multistage

amplifier of n similar valves and couplings can be drawn as Fig. 150(a), where u is the coupling impedance, i.e. the impedance, the anode and grid coupling impedances in parallel. The grid filament capacity of the first valve is omitted.

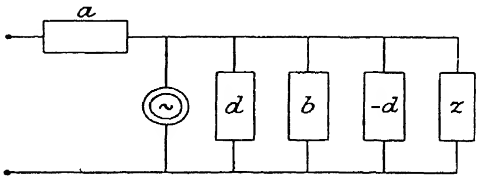


FIG. 151.

Let d be the impedance of ρ, C_3, u and C_1 in parallel so that

$$\frac{1}{d} = \frac{1}{\rho} + \frac{1}{u} + j\omega(C_1 + C_3);$$

and also let
$$\frac{1}{b} = \frac{1}{\rho} + j\omega C_3$$

The amplifier can now be redrawn in the simplified form shown in Fig. 150(b) and has been reduced to a repeated network of n units

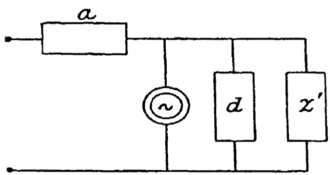


FIG. 152.

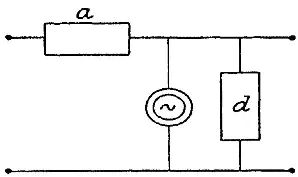


FIG. 153.

of which all but the last are similar. We can now redraw the last stage as Fig. 151 and if the impedance of b, $-d$ and z in parallel be z' so that

$$\frac{1}{z'} = \frac{1}{b} - \frac{1}{d} + \frac{1}{z}.$$

Fig. 151 may be redrawn as Fig. 152, and substituting this for the last stage in Fig. 150(b) the network reduces to n simple networks such as Fig. 153 terminated by z'.

Let the additional input impedance of the n stage amplifier be z_n. If there is only one valve, then from 64.3

$$z_1 = \frac{z'(a + d) + ad}{z'(1 + kd) + d} \qquad . \quad . \quad . \quad 65.1$$

and z_1 is thus obtained from z' by a homographic transformation; z_2 will of course be obtained by applying the same transformation to z_1, that is, by two successive applications to z'.

If now we choose three constants Z, θ, and ϕ so that

$$\left.\begin{aligned} \frac{\cosh(\theta + \phi)}{\cosh(\theta - \phi)} &= 1 + \frac{a}{d} \\ \frac{Z \sinh\theta}{\cosh(\theta - \phi)} &= a \\ \frac{\sinh\theta}{Z\cosh(\theta - \phi)} &= \frac{1 + kd}{d} \end{aligned}\right\} \qquad . \quad . \quad . \quad 65.2$$

equation 65.1 takes the form

$$z_1 = Z\frac{z'\cosh(\theta + \phi) + Z\sinh\theta}{z'\sinh\theta + Z\cosh(\theta - \phi)} \qquad . \quad . \quad 65.3$$

But this is of exactly the same form as 13.16 of Chapter II. when $n = 1$ and therefore, as therein shown,

$$z_n = Z\frac{z'\cosh(n\theta + \phi) + Z\sinh n\theta}{z'\sinh n\theta + Z\cosh(n\theta - \phi)} \qquad . \quad . \quad 65.4$$

and we have thus determined the additional input impedance of the n stage amplifier.

Let us now consider the receiving-end impedance, taking z' as the terminal impedance:—

From 64.5

$$y_1 = \frac{z'(a + d) + ad}{(1 - ak)d} \qquad . \quad . \quad . \quad 65.5$$

The numerator of the right-hand side of this equation is the same as that of the right-hand side of 65.1 and thus the equation may be written

$$y_1 = \frac{z'\cosh(\theta + \phi) + Z\sinh\theta}{\mu\cosh\phi}, \qquad . \quad . \quad 65.6$$

where μ is a new constant to be determined by

$$\frac{Z\sinh\theta}{\mu\cosh\phi} = \frac{a}{1 - ak}. \qquad . \quad . \quad . \quad 65.7$$

Equation 65.6 can be compared with the corresponding equation 13.21 and it will be seen that were the network passive we should have

$$\mu = 1$$

Following an argument exactly similar to that used in § 13 it can be shown that

$$y_n = \frac{z' \cosh (n\theta + \phi) + Z \sinh n\theta}{\mu^n \cosh \phi}, \quad . \quad . \quad . \quad 65.8$$

which differs only from 13.23 by the presence of μ^n.

The output voltage of the amplifier is the voltage across z, but this is also the same as the voltage across z'. Hence the output voltage

$$= \frac{ez'}{y_n} = \frac{ez' \mu^n \cosh \phi}{z' \cosh (n\theta + \phi) + Z \sinh n\theta},$$

and the voltage amplification is

$$\frac{z' \mu^n \cosh \phi}{z' \cosh (n\theta + \phi) + Z \sinh n\theta}. \quad . \quad . \quad . \quad 65.9$$

Equations 65.4 and 65.9 together with their auxiliary equations thus give a complete solution of the problem.

§ 66. The Multistage Amplifier with any Coupling.

If the coupling between valves is not direct it can always be represented by an equivalent Π shown as g, h, q in Fig. 154.

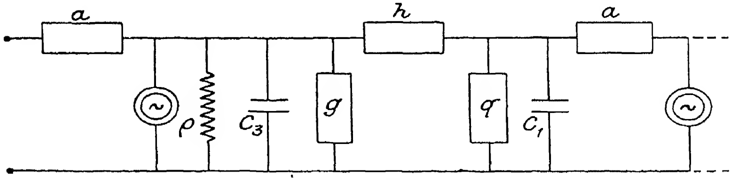


FIG. 154.

Let the impedance of ρ, C_3 and g in parallel be s so that

$$\frac{1}{s} = \frac{1}{\rho} + \frac{1}{g} + j\omega C_3,$$

and the impedance of q and C_1 in parallel be t so that

$$\frac{1}{t} = \frac{1}{q} + j\omega C_1.$$

Then an n stage amplifier can be drawn as in Fig. 155, and if we put

$$\frac{1}{z'} = \frac{1}{z} - j\omega C_1$$

the last stage becomes as Fig. 156, and the amplifier can then be considered as made of n networks such as Fig. 157 terminated by z'.

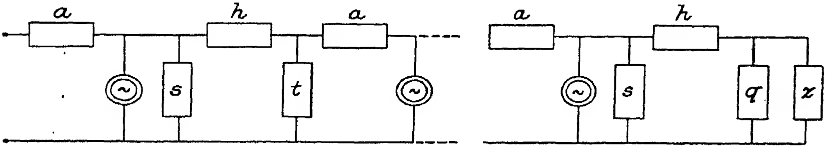


FIG. 155.

On working out the input impedance and receiving-end impedance of this network in a way similar to that used in § 64, we find that

$$z_1 = \frac{z'\{(h+t)(a+s)+as\}+t(as+ah+hs)}{z'\{(h+t)(1+ks)+s\}+t(h+hks+s)} \quad . \quad 66.1$$

and

$$y_1 = \frac{z'\{(h+t)(a+s)+as\}+t(as+ah+hs)}{st(1-ak)} . \quad 66.2$$

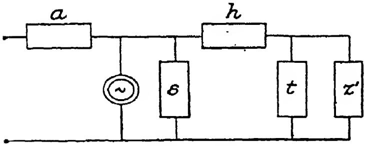


FIG. 156.

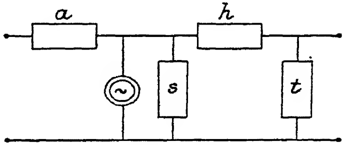


FIG. 157.

Following on the same lines as in the previous section, if Z, θ, ϕ, and μ are chosen so that

$$\left.\begin{aligned}
\frac{\cosh(\theta+\phi)}{\cosh(\theta-\phi)} &= \frac{(h+t)(a+s)+as}{t(as+ah+hs)} \\
\frac{Z\sinh\theta}{\cosh(\theta-\phi)} &= \frac{as+ah+hs}{h+hks+s} \\
\frac{\sinh\theta}{Z\cosh(\theta-\phi)} &= \frac{(h+t)(1+ks)+s}{t(h+hks+s)} \\
\frac{Z\sinh\theta}{\mu\cosh\phi} &= \frac{as+ah+hs}{s(1-ak)}
\end{aligned}\right\} \quad . \quad 66.3$$

the additional input impedance and the voltage ratio are determined by the same equations 65.4 and 65.9 as in the previous section—the values of Z, θ, ϕ and μ being of course obtained from 66.3.

§ 67. The Equivalent Valve Circuit when Grid Current Flows.

So far we have dealt with the valve with the assumption that there has been grid bias sufficient to prevent grid current. If,

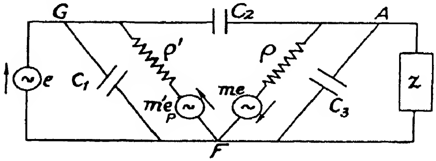


FIG. 158.

however, the grid bias is positive and grid current is flowing the equivalent circuit for the valve is more complicated and can be drawn as in Fig. 158 * where a fictitious source of voltage, proportional to e_P, the change in anode voltage, together with a series resistance ρ' has to be added between grid and filament; arrows have been put in the diagram to show the directions in which the voltages act.

Though apparently much more complicated than the equivalent circuit of § 64, it can be dealt with in exactly the same way.

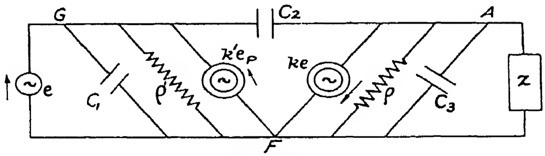


FIG. 159.

Replacing the fictitious sources of constant voltage by constant current sources we can replace Fig. 158 by Fig. 159, where $k' = m'/\rho'$, just as Fig. 148(b) replaced Fig. 148(a) in § 64.

We now require to determine the input impedance, the receiving-end impedance and the voltage amplification; in calculating the input impedance we can leave out C_1 and ρ' and consider only

* E. L. Chaffee, "Proc. I.R.E.," 17 Sept., 1929, page 1633.

the additional input impedance due to currents through C_2 and the constant current generator $k'e_P$.

If now we let

$$1/a = j\omega C_2$$
$$1/b = 1/\rho + j\omega C_3$$

then we have to consider the simple circuit shown in Fig. 160.

Assign circulating currents to the first three meshes and let d be the impedance of b and z in parallel so that

$$1/d = 1/b + 1/z.$$

Then the voltage across a added to the voltage across d must be equal to e, i.e.

$$ai_2 + di_3 = e. \qquad 67.1$$

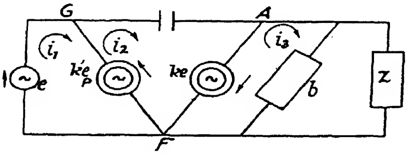


FIG. 160.

The sum of the currents flowing into the junction point A must be zero; hence

$$i_2 - i_3 - ke = 0. \qquad 67.2$$

Similarly the sum of the currents flowing into the junction-point G must be zero; hence

$$i_1 - i_2 + k'e_P = 0.$$

But e_P is the change of anode voltage and is equal to $- di_3$; hence substituting for e_P

$$i_1 - i_2 - k'di_3 = 0. \qquad 67.3$$

From these equations i_1, i_2 and i_3 can be obtained—

$$i_1 = \frac{e(1 + kd + k'd + adkk')}{(a + d)}$$
$$= e \cdot \frac{z\{1 + b(k + k' + akk')\} + b}{z(a + b) + ab} \qquad 67.4$$

$$i_3 = e \cdot \frac{1 - ak}{(a + d)} \qquad 67.5$$

Of these equations, the first, 67.4, gives the additional input impedance, while from the second it follows that the current through z, which is equal to $bi_3/(b + z)$, is

$$\frac{eb(1 - ak)}{(b + z)(a + d)}$$
$$= \frac{eb(1 - ak)}{z(a + b) + ab}. \qquad 67.6$$

This gives us the receiving-end impedance and therefore also the voltage amplification.

We could proceed now and develop the theory of the multi-stage amplifier with grid current; to do so requires only minor alterations in the procedure of the two previous sections.

It is remarkable (Chaffee, l.c.) about the equivalent circuits of Figs. 158 and 159 that they are of the same general form whether GF or AF are taken as input terminals, and thus by merely interchanging C_1, ρ' and m' with C_3, ρ and m respectively we can calculate the impedance of an amplifier viewed from the anode of the last valve with a given impedance connected between the grid and filament of the first valve.

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